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Practical illustration of the noetherization of an integro-differential operator A

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Abstract—The objective set in this research work is the construction of an illustrative, precise, and concise example of the construction of the noetherian theory of an integro-differential operator defined by an integral equation of third kind in certain types of functional spaces well defined. The motivations for carrying out such work lie in the aim of linking theoretical results to those obtained in practice by finding all the aspects of the theory in the construction of the said noetherian theory obtained in my previous research, relative to an integro-differential operator defined by an integral equation of the third kind in an analytical manner with well-defined concrete parameters. The illustration of the construction of the noetherian theory through the investigation of an integro-differential operator, defined by an integral equation of the third kind in a well-chosen functional space is the focal point of the work to be undertaken following the obtaining of our recent research results widely published in various references. To achieve this, we link the theoretical recipes to the practical ones by applying the different concepts and notions of the noetherian theory to a particular case of an operator *A* concretely defined analytically. We arrive at systematically defining the finite characteristic deficient numbers (α, β) of the integro-differential operator and, consequently, obtaining its finite index $Ind(A) = \chi(\mathrm{A}) = \alpha(\mathrm{A}) - \beta(\mathrm{A})$ confirming its noetherity. We base ourselves on the theoretical recipes obtained in our previous works to find them in the resolution of a concrete and detailed example, involving all the related results of the noetherian theory.

Keywords—noetherian theory, noetherization, third kind integral equation, singular linear integro-differential operator, deficient numbers, index of the operator, associated operator, associated space.

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I. INTRODUCTION

For several decades it has been well known and established that on the basis of noetherian theory widely devoted to various integro-differential operators defined by integral equations of the third kind studied in certain scientific research, several researchers have obtained the solvency conditions of the integral equations considered as a function of the orthogonality conditions of linear dependence of the solutions of the associated equation homogeneous in the associated space. Details on all these aspects can be found, for example in the following references [1], [2], [3], [4], [5], [6], [7], [8] and [9]. Note also that some particularities related to this aspect are mentioned in the paper [10] well illustrated within the research conducted in [11], [12], [13] and [14]. In the same way see also [15] and, we can refer the works [16] and [17] where have been investigated some classes of singular equations.

It should be noted that the process of construction of the noetherian theory which we also call noetherization for the integro-differential operators defined by certain types of integral equations of the third kind in certain specific functional spaces, leads us not only to indicate the conditions of the solvency of the integral equation considered but also in addition allows us to arrive at the establishment of the noetherity of the operators studied. Among many others, significant theoretical aspect and important scientific works have been done respectively within the references [18], [19], [20] and [21] on the basis of the theory indicated from [22] and [23]. We mention also approaches presented in the papers [24] and [25] some of main scientific ideas which were repeated and illustrated in the paper [26]. The steps taken in the work of the researchers who wrote the articles [27], [28] and [29] sufficiently show the application of noetherian theory to the operator studied and the establishment of the conditions of solvability of the integral equation defining it. In some specially studied cases, we were led to circumscribe specific approaches when we encountered difficulties related to the study of the solvency of the integral equations of the third type during the construction of the said theory for the integro-differential operators defined by such integral equations. On these particularities, one can refer to the paper [30] which was exactly completed by some results from the articles [31] and [32].

In our previous research carried out, it was necessary to choose well the necessary approach that allowed us to achieve the noetherization of the studied operator *A*. It should also be noted that several imminent scientists, well known researchers in the field of the noetherian theory have undertaken, depending on the cases investigated specific and necessary approaches during the process of noetherization of integro-differential operators in order to achieve the expected results. As approaches adopted, it is worth mentioning among others many, the method of normalization, the method of hypersingular integrals and the method of approximate inverse operators. Details on theoretical aspect with clearly presented investigations are located respectively in the books [1], [2] and also, within the scientific papers and works, [3], [4], [5], [6], [7], [8],[9], [10], [11], [12], [13],[14],[15],[16]and[17]. Specific approach was indicated when analyzing different situation investigated with the results obtained in [18], [19], [20] and [21]. In some particular aspects indicated, the results of the investigations are well illustrated based onto the general noetherian theory presented in the book [22]. Our researches undertaken in the past, the results of which are the subjects of the content of certain references in the bibliography, are also an illustration of such approaches adopted. The precise comparative and explanatory details relating thereto as well from our previously published results as from other scientits are located in the following references [23] and [24] on the basis of the theoretical receipt from the books [25] and [26]. In the theory from [27] and [28], illustrated through the research published in the papers [29], [30], [31] and [32], the method of the construction of the regularization of the noetherian operator is presented.

In this work, we build a special concrete, illustrative explanatory example of the establishment of the noetherian theory, finding all the demonstrated theoretical results, of the integro-differential operator A defined by a linear singular integral equation of the third kind of the following form: $(Ay)(x) = ax^p y^{(n)}(x) + bx^q y^{(n-1)}(x) + \int_0^1 k(x,t)\, y(t)dt = f(x); x \in [0,1]$.
with the unknown function $y(x) \in C^1[0,1]$, $f(x) \in C_0^{\{p\}}[0,1]$ and $k(x,t) \in C_0^{\{p\}}[0,1] \mathrm{X}\, \mathrm{C}\, [0,1]$ whose complete noetherization was carried out in our previous published papers presented also the specifity of the connexion with the regularization of the operator indicated the special case at the point 0.

Concretely, we illustrate the said theory in the following indicated Euler case situation example:

$$(A\,\varphi)(\mathrm{x}) = xy'(x) + \gamma y(x) + \lambda \int_0^1 xty(t)dt = f(x) \tag{1}$$

Through the realization of this concrete illustrative example within this work, we calculate and determine the deficient numbers $\alpha(A)$ and $\beta(A)$ and, also the index $\varkappa(A)$ of the operator A depending of the situation related to the relationship between the parameters of the studied operator.

Let us recall that in his related research, the author had been able to determine a necessary and sufficient condition for the solvability of several types of integral equations of the third kind in the class of Hölder function spaces. In a similar way and in the practical case we have invested ourselves in determining the necessary and sufficient condition for the solvability of equation (1) in the class of indicated functional spaces.

The work is organized as follows: firstly, we present in section 2 all the necessary preliminaries related to the various concepts and notions related to the well-known and widely used noetherian theory in several works dealing with questions of operator theory. Section 3 itself is devoted to the presentation of the constructed, concise, and clear illustrative example considered in the *Euler* case situation to be studied according to the different situations relating to the relationship between the parameters of the integral equation defining the operator. Thereafter, we summarize our steps and constructive ideas carried out in the investigations in section 4 which we call conclusion, followed by some recommendations necessary for the continuation of the future scientific work to be undertaken, set out in section 5.

II. PRELIMINARIES

In this part, we recall even before presenting in detail our main results, the following definitions and concepts well known and used in the theory of noetherian operators that we also find in a more general way from the references [1], [2], [3], [4], [5], [6], [7], [8] and [9] followed by the articles [10], [11] and [12]. On this matter, see also papers [13], [14], [15] and [16] where it has been presented various approaches for the noetherization of the considered operators. This being the case, we briefly review in some detail those important notions of the derivatives in the Taylor sense of a continuous function on a closed segment, which is widely used during the construction of the noetherian theory of certain integro-differential operators in some specific generalized functional spaces. They will appear gradually in the overall presentation of the work carried out dedicated to the concrete resolution of the problem posed.

Definition 2.1 We say that the function $\varphi(x) \in C[0,1]$ admits at the point $x = 0$ Taylor derivative up to the order $p \in \mathbb{N}$ if there exists recurrently for $k = 1,2,\dots,p,$ the following limits:

$$\varphi^{\{k\}}(0) = k! \lim_{x\to 0} x^{-k}\left[\varphi(x) - \sum_{j=0}^{k-1} \frac{\varphi^{\{j\}}(0)}{j!} x^j\right] \quad (2)$$

The class of such functions $\varphi(x)$ is designated by $C_0^{\{p\}}[0,1]$.

Let us also define a linear operator N^k on the space $C_0^{\{p\}}[-1,1]$ by the next formula:

$$(N^k\varphi)(x) = \frac{\varphi(x) - \sum_{j=0}^{k-1}\frac{\varphi^{\{j\}}(0)}{j!}x^j}{x^k}, k = 1,2,\dots,p. \quad (3)$$

One can easily verify the property $N^k = N^{k_1}N^{k-k_1}, 0 \le k_1 \le k, k, k_1 \in \mathbb{Z}_+$, where we put $N^0 = I$.

Related to the kernel of the integro-differential operator defined by *A*, let us mention that the kernel $k(x,t) \in C_0^{\{p\}}[0,1] X C[0,1]$, if and only if $k(x,t) \in C[0,1] X C[0,1]$ and admits Taylor derivatives according to the variable *x* at the point (0,*t*) whatever $t \in [0,1]$.

Let us move to the following important concept related to functional spaces and operators.

A) Associated operator and associated space.

Definition 2.2. We say that the Banach space $E' \subset E^*$ is called associated space with the space E, if the relationship
$|(f,\varphi)| \le c\|f\|_{E'}\|\varphi\|_E$
takes place for every $\varphi \in E, f \in E'$.

We note that the initial space E can be considered associated with the space E'. Let be noted $\mathcal{L}(E_1, E_2)$ the Banach algebra of all linear bounded operators from E_1 into E_2.

Definition 2.3. Let $E_j, j = 1,2$ two Banach spaces and E'_j their associated spaces. The operators $A \in \mathcal{L}(E_1, E_2)$ and $A' \in \mathcal{L}(E'_2, E'_1)$ are called associated, if and only if

$$(A'f, \varphi) = (f, A\varphi) \quad (4)$$

for all $f \in E'_2$ and $\varphi \in E_1$.

By defining the concept of associated space and associated operator, we used the references details from [1] and [2].

In the next following section, we recall and apply some important concepts in connection with the noetherian theory.

Definition 2.4 Let $A \in \mathcal{L}(E_1, E_2)$, we set $\alpha(A) = \dim\ker A$ – the number (of linearly independent) zero of the operator *A*; and $\beta(A) = \dim\mathrm{coker}A$ – the number of zero of the conjugate operator in the conjugate space.

Definition 2.5. Let $A \in \mathcal{L}(E_1, E_2)$ we denote and set $\chi(A) = \alpha(A) - \beta(A)$ – the index of the operator A.

In the case when $\alpha(A)$ and $\beta(A)$ are finite numbers, and the image of the operator *A* closed in E_2, then the operator *A* is called noether operator or simply noetherian operator.

It seems being suitable also to formalize the noetherity in terms of associated operator and associated space. Clear explanations with specific details can be found in various books connected with the noetherian theory.

Let us formalize the following important lemma.

Lemma 2.1 Let $E_j, j = 1,2$ two banach spaces and E'_j their associated spaces and, let $A \in \mathcal{L}(E_1, E_2)$ and $A' \in \mathcal{L}(E'_2, E'_1)$ be associated noetherian operators and more,

$$\alpha(A) = -\alpha(A'). \quad (5)$$

Then, for the solvability of the equation $A\varphi = f$ it is necessary and sufficient that $(f,\psi) = 0$ for all solutions of the homogeneous associated equation $A'\psi = 0$.

B) Pair of associated spaces.

We give the following definition.

Definition 2.6 Let $x_0 \in [0,1]$. Through $C_{x_0}^1[0,1]$ we represente the set of functions from $C^1[0,1]$ verifying the condition $\varphi(x_0) = 0$.

It is clear that, $C_{x_0}^1[0,1]$ is a Banach subspace in the space $C^1[0,1]$, if remarking, that for $\varphi_n(x_0) \in C_{x_0}^1[0,1]$ the convergence by norm $C^1[0,1]$ conducts $\varphi_n(x_0) \to \varphi(x_0), n \to \infty$, that, with respect to $\varphi_n(x_0) = 0$ for all $n \in \mathbb{N}$ leads us to $\varphi(x_0) = 0$.

Now let us state this important necessary lemma within this work.

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Lemma 2.2 The space $C^1_{x_0}[0,1]$ is associated to the space $C[0,1]$.

Proof: Obvious.

Definition 2.7 Let the $\delta^{\{k\}}(x)$ – *k-th* Taylor derivative of Dirac delta function be understood and defined in the following way:

$(\delta^{\{k\}}(x), \varphi(x)) = \int_{-1}^{1} \delta^{\{k\}}(x)\,\varphi(x)dx = (-1)^k \varphi^{\{k\}}(0)$

Lemma 2.3. Let $p \in \mathbb{N}, s \in \mathbb{Z}_+$. If $\varphi(x) \in C_0^{\{s\}}[0,1]$ then, $x^p\varphi(x) \in C_0^{\{p+s\}}[0,1]$, and the formula holds

$$\left(x^p\varphi(x)\right)^{\{j\}}(0) = \begin{cases} 0, \; j = 0,1,\ldots,p-1, \\ \frac{j!}{(j-p)!}\varphi^{\{j-p\}}(0), j = p,\ldots,p+s. \end{cases} \tag{6}$$

Proof. Obvious.

From the previous lemmas 2.1 and 2.2, it follows the next important lemma.

Lemmas 2.1 and 2.2 imply the next lemma.

Lemma2.4.

Let

$f(x) \in C_0^{\{p\}}[0,1], p \in \mathbb{N} \; and \; f(0) = \ldots.. = f^{\{r-1\}}(0) = 0, 1 \leq r \leq p.$

Then $\frac{f(x)}{x^r} \in C_0^{\{p-s\}}[0,1]$.

It is also convenient to use an equivalent definition for the norm in the space $C_0^{\{p\}}[0,1]$:

$$\|\varphi\|^1_{C_0^{\{p\}}[0,1]} = \sum_{j=0}^{p} \|N^j\varphi\|_{C[0,1]} \tag{7}$$

Lemma2.5. The operator $N^p: C_0^{\{p\}}[0,1] \longrightarrow C[0,1]$ has the following properties:

1. N^p is bounded, and $\|N^p\varphi\|_{C[0,1]} \leq \|\varphi\|_{C_0^{\{p\}}[0,1]}$;

2. N^p is right invertible;

3. $\alpha(N^p) = p$, where $\alpha(N^p)$ is the dimension of the null subspace for N^p.

For the proof of the previous lemma with full details we can refer to the general theory from [1] and [2].

Next, in the following section, we will present the concrete model of the situation of an already noetherized integro-differential operator to be studied for its noetherity (find the results of the theory) respectively in the functional spaces thus chosen, by bringing out from these searches for the solvency conditions of the integral equation of the third kind having defined it.

III. MAIN GLOBAL IMPORTANT RESULTS

In this section, which contains the main results of our research, we focus on the application of the theory to the practical case to be investigated. In other words, we seek the solvency conditions of the integral equation of the third kind defining the integro-differential operator in parallel with the construction of the noetherian theory of said operator.

The Euler case illustrative situation.

We return to the investigated case of the equation (01) in the case when the $min(p,q) = min(1,0) = 0$.

Namely, we investigate the solvency and noetherity of the integral equation defined by the formula (1) of the following form:

$$(A\,\varphi)(x) = xy'(x) + \gamma y(x) + \lambda\int_0^1 xty(t)dt = f(x)$$

where the function $f(x) \in C_0^{\{1\}}[0,1]$, $y(x) -$ the unknown function is taken from the space $C^1[0,1]$. $k(x,t) = \lambda xt \in C_0^{\{1\}}[0,1] X C[0,1]$, $\gamma \in \mathbb{R}$.

Let us denote through

$$c = \int_0^1 ty(t)dt, \tag{8}$$

Then the equation (1) has the following form:

$$xy'(x) + \gamma y(x) + \lambda xc = f(x), \tag{9}$$

Or the same as

$$xy'(x) + \gamma y(x) = f(x) - \lambda xc = f_1(x) \in C_0^{\{1\}}[0,1]. \tag{10}$$

1) Let at the beginning $\gamma > -1, \gamma \neq 0$. Then, the unknown function $y(x)$ is defined by the formula:

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$$y(x) = x^{-\gamma}\int_0^x t^{\gamma}(Nf_1)(t)dt + \frac{f_1(0)}{\gamma} = x^{-\gamma}\int_0^x t^{\gamma} N[f(x) - \lambda xc](t)dt + \frac{f_1(0)}{\gamma} = x^{-\gamma}\int_0^x t^{\gamma}(Nf)(t)dt - x^{-\gamma}\lambda c\int_0^x t^{\gamma}dt + \frac{f(0)}{\gamma} = x^{-\gamma}\int_0^x t^{\gamma}(Nf)(t)dt - \lambda c\frac{x}{\gamma+1} + \frac{f(0)}{\gamma}. \tag{11}$$

Now let us define $c = \int_0^1 sy(s)ds$, where $y(s)$ is defined by the formula (11).

$$c = \int_0^1 ss^{-\gamma}ds\int_0^s t^{\gamma}(Nf)(t)\,dt - \frac{\lambda c}{\gamma+1}\int_0^1 s^2ds + \frac{f(0)}{\gamma}\int_0^1 sds = \int_0^1 s^{-\gamma+1}\,ds\int_0^s t^{\gamma}(Nf)(t)dt - \frac{\lambda c}{3(\gamma+1)} + \frac{f(0)}{2\gamma} \tag{12}$$

Now changing the order of the integration, we have

$$c = \int_0^1 t^{\gamma}(Nf)(t)dt\int_t^1 s^{-\gamma+1}\,ds - \frac{\lambda c}{3(\gamma+1)} + \frac{f(0)}{2\gamma} = \int_0^1 t^{\gamma}(Nf)(t)\left(\frac{1}{2-\gamma} - \frac{t^{-\gamma+2}}{2-\gamma}\right)dt - \frac{\lambda c}{3(\gamma+1)} + \frac{f(0)}{2\gamma} = \frac{f(0)}{2\gamma} - \frac{\lambda c}{3(\gamma+1)} + \int_0^1 (Nf)(t)\left(\frac{t^{\gamma} - t^2}{2-\gamma}\right)dt \tag{13}$$

Making some computations we arrive to:

$$c\left(1 + \frac{\lambda}{3(\gamma+1)}\right) = \frac{f(0)}{2\gamma} + \int_0^1 \left(\frac{t^{\gamma} - t^2}{2-\gamma}\right)(Nf)(t)\,dt \tag{14}$$

a) Let at the beginning $\left(1 + \frac{\lambda}{3(\gamma+1)}\right) \neq 0$.

Then we can find c in a unique way and this constant is defined by the following way:

$$c = \frac{3(\gamma+1)}{3(\gamma+1)+\lambda}\left[\frac{f(0)}{2\gamma} + \frac{1}{2-\lambda}\int_0^1 t^{\gamma-1}[f(t) - f(0)]\,dt - \frac{1}{2-\gamma}\int_0^1 t[f(t) - f(0)]\,dt\right] = \frac{3(\gamma+1)}{3(\gamma+1)+\lambda}\left[\frac{f(0)}{2\gamma} + \frac{1}{2-\lambda}\int_0^1 (t^{\gamma-1} - t)f(t)\,dt + \frac{f(0)}{2-\gamma}\left(\frac{1}{2} - \frac{1}{\gamma}\right)\right] = \frac{3(\gamma+1)}{3(\gamma+1)+\lambda}\left[\frac{f(0)}{2\gamma} + \frac{1}{2-\lambda}\int_0^1 (t^{\gamma-1} - t)f(t)\,dt + \frac{f(0)}{2-\gamma}\frac{\gamma-2}{2\gamma}\right] = \frac{3(\gamma+1)}{3(\gamma+1)+\lambda}\cdot\frac{1}{2-\lambda}\int_0^1 (t^{\gamma-1} - t)f(t)\,dt \tag{15}$$

Therefore, we can finally find $y(x)$ and it is defined the only way by the formula which, when $\gamma > 0$ has the form:

$$y(x) = x^{-\gamma}\int_0^x t^{\gamma}(Nf)(t)dt - \frac{3\lambda(\gamma+1)x}{(3(\gamma+1)+\lambda)(\gamma+1)}\cdot\frac{1}{2-\lambda}\int_0^1 (t^{\gamma-1} - t)f(t)\,dt + \frac{f(0)}{\gamma}. \tag{16}$$

When $-1 < \gamma < 0$ it follows to set c in the form (14) without making the transformation (15).

Therefore, the equation (1) is solvable under any right hand side $f(x) \in C_0^{\{1\}}[0,1]$, and its solution is defined by the formula (16).

The operator A is therefore noetherian with the index $\chi(A) = 0$ and deficient numbers $(0,0)$, and therefore, $A: C^1[0,1] \longrightarrow C_0^{\{1\}}[0,1]$ is invertible.

b) Let now $\left(1 + \frac{\lambda}{3(\gamma+1)}\right) = 0$, i.e. $\lambda = -3(\gamma+1)$, then we obtain

$$\frac{f(0)}{2\gamma} + \int_0^1 (Nf)(t)\left(\frac{t^{\gamma} - t^2}{2-\gamma}\right)dt = 0. \tag{17}$$

Therefore it is necessary

$$f(0) = -2\gamma\int_0^1 (Nf)(t)\left(\frac{t^{\gamma} - t^2}{2-\gamma}\right)dt \tag{18}$$

Under the accomplishment of the condition of solvency (18) with respect to the form of the solution by the formula (12), we obtain:

$$y(x) = x^{-\gamma}\int_0^x t^{\gamma}(Nf)(t)dt + \frac{f(0)}{\gamma} + cx, \tag{19}$$

where c — any arbitrary constant.

Therefore and consequently the equation (1) is solvable under accomplishment of one condition on the right hand side $f(x)$, of the form (18) and the solution $y(x)$ is defined by the formula (19), where $c -$ any arbitrary constant . The operator A is noetherian with the index $\chi(A) = 0$ and deficient numbers $(1,1)$.

II) Now let $\gamma = 0$. Then the equation (1) is equivalent to the equation:

$$xy'(x) + \lambda \int_0^1 xty(t)dt = f(x), \quad (20)$$

where the unknown function is from the class $C^1[0,1]$, $f(x) \in C_0^{\{1\}}[0,1]$.

Designating $c = \int_0^1 ty(t)dt$ we have:

$$xy'(x) = f(x) - \lambda c = f_1(x) \in C_0^{\{1\}}[0,1].$$

For the solvency of this equation, it is necessary that $f_1(0) = f(0) = 0$.

Under accomplishment of this condition the solution of the equation (20) in the space $C^1[0,1]$ has the following form:

$$y(x) = \int_0^x N[f(t) - \lambda xc](t)dt + c_1, 0 \le x \le 1; \quad (21)$$

or the same as

$$y(x) = \int_0^x (Nf)(t)dt - \lambda xc + c_1, \quad (22)$$

where $c_1 -$ an arbitrary constant. Now let us determine $c = \int_0^1 sy(s)ds$, where $y(s)$ is defined by the formula (22).

$$c = \int_0^1 sds \int_0^s (Nf)(t)dt - \lambda c \int_0^1 s^2 ds + \frac{c_1}{2} = \int_0^1 (Nf)(t)dt \int_t^1 sds - \frac{\lambda}{3}c + \frac{c_1}{2} = \frac{c_1}{2} + \int_0^1 (Nf)(t)\left[\frac{1}{2} - \frac{t^2}{2}\right]dt - \frac{\lambda}{3}c. \quad (23)$$

For the value of c we have:

$$c\left(1 + \frac{\lambda}{3}\right) = \frac{c_1}{2} + \int_0^1 (Nf)(t)\left[\frac{1}{2} - \frac{t^2}{2}\right]dt. \quad (24)$$

a) Let $1 + \frac{\lambda}{3} \neq 0$, then c is defined the only way from (24) and

$$c = \frac{3}{\lambda+3}\left[\frac{c_1}{2} + \int_0^1 (Nf)(t)\left[\frac{1}{2} - \frac{t^2}{2}\right]dt\right] \quad (25)$$

Consequently, we can finally find the solution $y(x)$ which is defined by the following:

$$y(x) = \int_0^x (Nf)(t)dt - \frac{3\lambda x}{\lambda+3}\left[\frac{c_1}{2} + \int_0^1 (Nf)(t)\left(\frac{1}{2} - \frac{t^2}{2}\right)dt\right] + c_1 \quad (26)$$

where $c_1 -$ an arbitrary constant.

Consequently, the equation (1) is solvable under the accomplishment of one condition onto the right-hand side $f(x) \in C_0^{\{1\}}[0,1]$ of the form $f(0) = 0$ and the solution $y(x)$ is defined by the formula (26) with accuracy up to one constant.

Therefore, the operator A is noetherian with the index $\chi(A) = 0$ and the deficient numbers $(1,1)$.

The solution of the homogeneous equation (26) as it is seen will be:

$$y_0(x) = c_1\left(1 - \frac{3\lambda x}{2(\lambda+3)}\right).$$

b) Let now $1 + \frac{\lambda}{3} = 0$, i.e. $\lambda = -3$. Then from (24) it is clear that:

$$\frac{c_1}{2} + \int_0^1 (Nf)(t)\left(\frac{1}{2} - \frac{t^2}{2}\right)dt = 0 \quad (27)$$

from what:

$$\int_0^1 (Nf)(t)\left(\frac{1}{2} - \frac{t^2}{2}\right)dt = -\frac{c_1}{2} \quad (28)$$

Or the same as

$$\int_0^1 (Nf)(t)(1 - t^2)dt = -c_1. \quad (29)$$

Consequently, with respect to the form of the solution by the formula (22) we reach the expression for $y(x)$ of the form with the supposition that $f(0) = 0$.

$$y(x) = \int_0^x (Nf)(t)dt - \int_0^1 (Nf)(t)(1 - t^2)dt - cx \quad (30).$$

Here $c -$ already is an arbitrary constant. Therefore, under $\lambda = -3$, A is noetherian operator with the index $\chi(A) = 0$ and deficient numbers $(1,1)$.

Remark that by virtue $f(0) = 0$ we can give to the solution (30) a more simple form.

$$y(x) = -\int_x^1 \frac{f(t)}{t}dt + \int_0^1 tf(t)dt - cx, \quad (31)$$

where $y_0(x) = cx-$ the solution of the homogeneous equation (20).

III) Let now $\gamma < -1$. In this case, for the solution of the equation

$$xy'(x) + \gamma y(x) = f_1(x) = f(x) - \lambda xc \in C_0^{\{1\}}[0,1]$$

we have:

$$y(x) = -x^{-\lambda}\int_x^1 t^\gamma (Nf_1)(t)dt + \frac{f_1(0)}{\gamma} + c_1x^{-\lambda} = -x^{-\lambda}\int_x^1 t^\gamma[(Nf)(t) - \lambda c]dt + \frac{f(0)}{\gamma} + c_1x^{-\lambda} = -x^{-\lambda}\int_x^1 t^\gamma (Nf)(t)dt + \lambda cx^{-\lambda}\frac{t^{\gamma+1}}{\gamma+1} + c_1x^{-\lambda} + \frac{f(0)}{\gamma}. \tag{32}$$

From where it follows after computations the next result:

$$y(x) = -x^{-\lambda}\int_x^1 t^\gamma (Nf)(t)dt + \frac{\lambda c}{\gamma+1}x^{-\lambda} - \frac{\lambda c}{\gamma+1}x + c_1x^{-\lambda} + \frac{f(0)}{\gamma}. \tag{33}$$

Next, let put $c = \int_0^1 sy(s)ds$. We have

$$c = -\int_0^1 s^{-\gamma+1}\left[\int_0^x t^\gamma (Nf)(t)dt\right]ds - \frac{\lambda c}{\gamma+1}\int_0^1 s^2 ds + \left(c_1 + \frac{\lambda c}{\gamma+1}\right)\int_0^1 s^{-\gamma+1}ds + \frac{f(0)}{2\gamma} = -\int_0^1 t^\gamma (Nf)(t)dt\int_0^t s^{-\gamma+1}ds - \frac{\lambda c}{3(\gamma+1)} - \left(c_1 + \frac{\lambda c}{\gamma+1}\right)\frac{1}{\gamma-2} + \frac{f(0)}{2\gamma},$$

where

$$c\left[1 + \frac{\lambda}{3(\gamma+1)} + \frac{\lambda}{(\gamma-2)(\gamma+1)}\right] = \frac{f(0)}{2\gamma} + \frac{1}{\gamma}\int_0^1 (Nf)(t)dt - \frac{c_1}{\gamma-2},$$

or the same as:

$$c\left[1 + \frac{\lambda}{3(\gamma-2)}\right] = \frac{f(0)}{2\gamma} + \frac{1}{\gamma}\int_0^1 (Nf)(t)dt - \frac{c_1}{\gamma-2} \tag{34}$$

a) $\lambda \neq -3(\gamma-2)$. In this case the constant c is defined from (34) and the solution is expressed by the following:

$$y(x) = -x^{-\lambda}\int_x^1 t^\gamma (Nf)(t)dt + \frac{f(0)}{\gamma} + \frac{\lambda(x^{-\lambda}-x)}{\gamma+1}\cdot -\frac{3(\gamma-2)}{\lambda+3(\gamma-2)}\left[\frac{f(0)}{2\gamma} + \frac{1}{\gamma}\int_0^1 (Nf)(t)dt - \frac{c_1}{\gamma-2}\right] + c_1x^{-\lambda} = -x^{-\lambda}\int_x^1 t^\gamma (Nf)(t)dt + \frac{f(0)}{\gamma}\left(1 + \frac{3\lambda(\gamma-2)(x^{-\lambda}-x)}{2(\gamma+1)[\lambda+3(\gamma-2)]}\right) + \frac{3\lambda(\gamma-2)(x^{-\lambda}-x)}{\gamma(\gamma+1)[\lambda+3(\gamma-2)]}\int_0^1 (Nf)(t)dt + c_1\left[x^{-\lambda} - \frac{3\lambda(x^{-\lambda}-x)}{(\gamma+1)[\lambda+3(\gamma-2)]}\right], \tag{35}$$

where $c_1 -$ an arbitrary constant and

$$y_0(x) = c_1\left[x^{-\lambda} - \frac{3\lambda(x^{-\lambda}-x)}{(\gamma+1)[\lambda+3(\gamma-2)]}\right] \tag{36}$$

The solution of the homogeneous of the equation (1).
Now let us verify that in fact we get the following:
Let $c_1 = 1$, we have

$$y_0'(x) = -\gamma x^{-\gamma-1} - \frac{3\lambda}{(\gamma+1)[\lambda+3(\gamma-2)]}[-\gamma x^{-\gamma-1} - 1],$$

so that

$$xy_0'(x) + \gamma y_0(x) = -\gamma x^{-\gamma} - \frac{3\lambda}{(\gamma+1)[\lambda+3(\gamma-2)]}[-\gamma x^{-\gamma} - x] + \gamma x^{-\lambda} - \frac{3\lambda\gamma}{(\gamma+1)[\lambda+3(\gamma-2)]}[x^{-\lambda} - x] = \frac{3\lambda x}{\lambda+3(\gamma-2)}.$$

So that

$$xy_0'(x) + \gamma y_0(x) = \frac{3\lambda x}{\lambda+3(\gamma-2)} \tag{37}$$

Let us now calculate:

$$\lambda x\int_0^1 ty_0(t)dt = \lambda x\int_0^1\left(t^{-\gamma} - \frac{3\lambda(t^{-\gamma}-t)}{(\gamma+1)[\lambda+3(\gamma-2)]}\right)tdt = \frac{\lambda x}{2-\gamma}\left[1 - \frac{3\lambda}{(\gamma+1)[\lambda+3(\gamma-2)]}\right] + \lambda x\frac{\lambda}{(\gamma+1)[\lambda+3(\gamma-2)]} = \frac{\lambda x}{2-\gamma} + \frac{\lambda x}{(\gamma+1)[\lambda+3(\gamma-2)]}\left(\lambda - \frac{3\lambda}{2-\gamma}\right) = \frac{\lambda x}{2-\gamma} - \frac{\lambda^2 x}{(2-\gamma)[\lambda+3(\gamma-2)]} = \frac{\lambda x}{2-\gamma}\left(1 - \frac{\lambda}{\lambda+3(\gamma-2)}\right) =$$

$$= \frac{\lambda x}{2-\gamma}\cdot\frac{3(\gamma-2)}{\lambda+3(\gamma-2)} = -\frac{3\lambda x}{\lambda+3(\gamma-2)} \tag{38}$$

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The unification of (37) and (38) gives the next result:

$$Ay_0(x) = xy_0'(x) + \gamma y_0(x) + \lambda x \int_0^1 ty_0(t)dt = 0.$$

So that in this case, the operator $A\colon\ C^1[0,1] \longrightarrow C_0^{\{1\}}[0,1]$ is noetherian with the index $\chi(A) = +1$ and deficient numbers $(1,0)$.

b) $\lambda = -3(\gamma - 2)$. In this case, from (34) it is not possible to define the unknown constant C and it will be necessary to require that the right hand side (34) will turn into zero. In this case, the homogeneous equation has the solution:

$$y_0(x) = c(x^{-\lambda} - x), \tag{39}$$

where $C -$ an arbitrary constant. With respect to (39) the solution in the form (after the changing C_1 with regard to (34))

$$y(x) = c(x^{-\gamma} - x) - x^{-\gamma}\int_x^1 t^{\gamma}(Nf)(t)dt + \frac{f(0)}{\gamma} + (\gamma - 2)\left(\frac{f(0)}{2\gamma} + \frac{1}{\gamma}\int_0^1 (Nf)(t)dt\right)x^{-\lambda}. \tag{40}$$

i) Verify, that $y_0(x) = c(x^{-\gamma} - x)$ is the solution of the homogeneous equation. In fact,

$$y_0{}'(x) = c(-\gamma x^{-\gamma-1} - 1),$$
$$xy_0'(x) + \gamma y_0(x) = -(1+\gamma)xc.$$

In the other side:

$$\lambda x \int_0^1 ty_0(t)dt = \lambda cx \int_0^1 t(t^{-\gamma} - t)dt = \lambda cx\left(-\frac{1}{\gamma - 2} - \frac{1}{3}\right).$$

For that,

$$Ay_0(x) = xy_0'(x) + \gamma y_0(x) + \lambda x\int_0^1 ty_0(t)dt = -(1+\gamma)cx + \lambda cx\left(-\frac{1}{\gamma-2} - \frac{1}{3}\right) = -cx\left(1 + \gamma + \frac{\lambda(1+\gamma)}{3(\gamma-2)}\right) = 0 \tag{41}$$

by virtue of the supposition, $\lambda = -3(\gamma - 2)$.

ii) Now let us turn to the particular solution

$$y_p(x) = -x^{-\gamma}\int_x^1 t^{\gamma}(Nf)(t)dt + \frac{f(0)}{\gamma} + (\gamma - 2)\left(\frac{f(0)}{2\gamma} + \frac{1}{\gamma}\int_0^1 (Nf)(t)dt\right)x^{-\gamma}, \tag{42}$$

$$y_p'(x) = \gamma x^{-\gamma-1}\int_x^1 t^{\gamma}(Nf)(t)dt + (Nf)(x) - (\gamma - 2)\gamma x^{-\gamma-1}\left(\frac{f(0)}{2\gamma} + \frac{1}{\gamma}\int_0^1 (Nf)(t)dt\right). \tag{43}$$

$$xy_p'(x) + \gamma y_p(x) = x(Nf)(x) + f(0) = f(x).$$

Next, we find

$$I = \lambda x\int_0^1 ty_p(t)dt = \lambda x\int_0^1 t\left[-t^{-\gamma}\int_s^1 s^{\gamma}(Nf)(s)ds + \frac{f(0)}{\gamma} + ut^{-\gamma}\right]dt, \tag{44}$$

where $u = (\gamma - 2)\left(\frac{f(0)}{2\gamma} + \frac{1}{\gamma}\int_0^1 (Nf)(t)dt\right)$.

Consequently,

$$I = -\lambda x\int_0^1 s^{\gamma}(Nf)(s)ds\int_0^s t^{-\gamma-1}dt + \frac{\lambda x f(0)}{2\gamma} - \lambda x u\frac{1}{\gamma - 2} = \frac{\lambda x}{\gamma}\int_0^1 (Nf)(s)ds + \frac{\lambda x f(0)}{2\gamma} - \lambda x\left[\frac{f(0)}{2\gamma} + \frac{1}{\gamma}\int_0^1 (Nf)(s)ds\right] = 0. \tag{45}$$

So that, $Ay_p(x) = f(x)$.

Therefore, in this case as previously the operator $A\colon\ C^1[0,1] \longrightarrow C_0^{\{1\}}[0,1]$ is noetherian with the index $\chi(A) = +1$ and deficient numbers $(1,0)$.

IV. CONCLUSION

This practical exercise resolved in the form of scientific activity based on the application of the noetherian theory well established in our previous work, shows in all aspects, the real interconnection between theory and practice. The integral equation of the third kind of general form, defining the integro-differential operator *A*, having been the subject of our previous investigations for the construction of the noetherian theory (noetherization) has been investigated in parallel for its solvency.

We have also completely found all the results of the theoretical aspects mentioned in relation to said theory in this illustrative example consolidating the interconnection between the two areas: theory and practice.

The results obtained would consolidate future research to be carried out within the framework of the finite-dimensional extension of said operator, by adding to the initial space $C^1[0,1]$ and separately, a space of a finite linear combination of the Dirac delta distributions as well as its successive derivatives in the Taylor sense

$\left\{\sum_{k=0}^{m}\alpha_k\delta^{\{k\}}(x)\right\}$, a space of principal parts of the improper integrals in the Hadamard sense $\left\{\sum_{j=1}^{n}\beta_j F.p\frac{1}{x^j}\right\}$, and, in last position the direct sum of the two previous functional spaces and the initial space considered i.e

$$C^1[-1,1]\oplus\left\{\sum_{k=0}^{m}\alpha_k\delta^{\{k\}}(x)\right\}\oplus\left\{\sum_{j=1}^{n}\beta_j F.p\frac{1}{x^j}\right\}.$$

V. RECOMMENDATIONS

In connection with what has already been done in our previous research on the theoretical level, the results of which have been widely published, there is reason to wonder about the applicability of the said theory on concrete cases to be studied to reflect the latter. The illustrative examples always make it possible to better understand the study carried out in a theoretical way and sometimes, facilitates the vision of the applications on the practical level with the example of the resolution of the problems of transport whose solutions are presented in the form of the elements of the spaces extensions of the integro-differential operators of the third kind, defined by the integral equations of the third kind which have been the object of some important research carried out. Through this approach, we should systematically find through these illustrative examples, a clear and concise interconnection, once again, between theory and practice. Naturally, it will be necessary in the same direction, to fully cover the subject, to achieve the generalization not only of our theoretical results of the investigation on such a subject of noetherization of these types of integro-differential operators defined by an integral equation of the third kind of higher order of the following type:

$$(Ay)(x)=ax^p y^{(n)}(x)+bx^q y^{(n-1)}(x)+\int_{-1}^{1}K(x,t)y(t)dt=f(x);x\in[-1,1]$$

but also, get there to build edifying illustrative examples.

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Contribution of individual authors to the creation of a scientific article (ghostwriting policy).

Abdourahman HAMAN ADJI had set the problem investigated and proposed completely the methodology of the research conducted within the whole work with all computations.

SHANKISHVILI Lamara Dmitrievna verified also some computations realized within the whole work, established links with some papers related to noetherian theory of integro-differential operators.

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Conflicts of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.