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Noetherization Theory for a Singular Linear Differential Operator of Higher Order

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Abstract—The main objective set in this research is the construction of noetherian theory for a singular linear integro-differential operator *L* defined by a linear singular differential equation of higher order in a specific functional space well chosen to achieve the goal. It should be emphasized that the case where $n = 1$ has been completely studied in the two situations separately when $p = 1$ and $p \geq 2$. Our previous various published research was related to this topic. The methodology adopted on a case-by-case basis, and depending on the values and sign of the parameter $\gamma \in \mathbb{R}$, leads us to solve the linear differential equation studied with a well-known second specific right-hand side $f(x) \in C_0^{\{p\}}[-1,1]$, systematically identifying the conditions solvency.

One of the major difficulties arising in this work, apart from those related to the quantitative techniques of solving the differential equation defining the considered integro-differential operator, is the construction of the continuity of the regularizers by starting from the smallest segment [0,1] to completely cover the whole closed interval [-1,1]. We rely on the approaches and methods built by the researcher Yurko V.A to achieve the noetherity of the considered operator.

This takes us straight to the investigation and construction of the noetherity of the operator *L*. Finally, depending on each case, we evaluate and calculate the deficient numbers and the index of the operator considered in various situations, relative to the parameter $\gamma \in \mathbb{R}$.

Keywords—noetherian theory, third kind integral equation, singular linear integro-differential operator, deficient numbers, index of the operator, associated operator, associated space, noetherian theory, noetherization.

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I. INTRODUCTION

The construction of the noetherian theory for the integro-differential operators defined by certain forms of integral equations of the third type is widely illustrated by the investigations carried out in certain scientific research works, [1], [2], [3], [4], [5] and [6]. On this matter one can also refer to the papers [7], [8], [9], where the author investigated the solution of a linear integral equation of third kind. Noetherian theory has been devoted to specific types of integral operators studied in the papers [10], [11] and [12]. However, let us mention that the main difficulty we face in the case of the investigation and study of the solvability of integral equations of the third type during the construction of the noetherian theory for the operators defined by such integral equations lies in the choice of necessary approaches and methods leading to the expected goal. It should be noted that in several works carried out by researchers related to the said subject from the papers, [13], [14], [15], [16] remarkably well-illustrated from the general theory in [17] and [18], the application of the normalization method, the method of hypersingular integrals and also the method of approximate inverse operators made it possible to approach the investigations of the integral equations of the third type in the space of continuous functions and distributions.

These different approaches and methods make it possible to clearly define and pose the problem when carrying out studies relating to integral equations of the first type, or non-Fredholm integral equations of the second type which define the operators thus considered. Let us mention the problem of searching the solutions of a nonlinear integral equation for high energy scattering investigated in the paper [19]. At the same time, let us also underline that it has been carried out the studies of the inverse problem of the spectral analysis of the theory of linear multigroup neutron transport in plane geometry in the work devoted in the scientific article titled Transp. Theory Stat. Phys. 29 (2000), No. 6, 711-722.

Recall that one researcher carried out the investigations in the space of Holder functions on a non-homogeneous linear integral equation with coefficient *cos x* and was able to identify the necessary and sufficient conditions for the solvability of the equation considered under certain indicated hypotheses on its core. On the other hand, let us also note that he was able to succeed in constructing the solution to the equation studied analytically, using the application of Fredholm's theory and that of singular integral equations. In relation to said investigations one can refer and read more details from some papers devoted to this aspect.

Let us also remember the work in which the researcher carried out the investigations in his article relating to Inverse problems for homogeneous transport equations, I. The results of the studies undertaken in the one-dimensional case have been published. Inverse Problems 16 (2000), No. 4, 997-1028. Relative to our previous investigations, we have already carried out the construction of the noetherian theory of an integro-differential operator defined by a singular linear integral equation of the third type of the following form:

$$(A\varphi)(x) = x^p\varphi'(x) + \int_{-1}^{1} K(x,t)\varphi(t)dt = f(x); x \in [-1,1]$$

with the unknown function $\varphi \in C_{-1}^{1}[-1,1]$, the second right-hand side $f(x) \in C_0^{\{p\}}[-1,1]$ and the kernel $K(x,t) \in C_0^{\{p\}}[-1,1] \text{ X C } [-1,1]$.

In the investigations carried out concerning the third kind integral equations, defining these integro-differential operators, the development of several diverse approaches is necessary to achieve the noetherization of the latter. In the case of the investigations carried out concerning the third kind singular integral equations in functional spaces and distributions, one can mention the realization of the methods mentioned above and presented globally in the works [1], [2] and [3] from one side. On the other side remark that all necessary details are localized in the references [4], [5] and also presented with some particularities in the papers [6], [7], [8], [9] and [10]. In the same way, we mention special approaches adopted when establishing the noetherian theory in the papers [11], [12] without forgetting to evoke these aspects studied in the references [13], [14], [15] and [16]. In these works, the particular specificity of the operator *L* playing the role of the principal part of the integro-differential operator is a multiplicative operator by a continuous function that vanishes in a finite or countable set of points on the investigation interval.

From the general theory presented in [17], [18] it clearly follow all the explanations and other necessary details of such research. Remark that more deeply have been done investigations by the authors of the papers [19] and [20] when they studied concrete physical phenomenon situations to illustrate this noetherian theory on some integral equations. A special approach in connexion when realizing the establishment of the noetherization of the finite dimensional extension of a noetherian operator has been described completely in the paper [21] and also based on the theoretical illustrative aspect from [22]. Specific ideas presented in the papers [23], [24], [25], [26] and [27] illustrated various serious scientific results obtained by the authors related to the conditions of solvency of some singular integral operators in different functional spaces. In the same direction, let us note that two authors have presented in their research the noetherian theory of a singular integral functional operator of finite order in the continuous case by making a substantial introduction relating to this particular case of investigation in their article [28]. In order to approach a generalization of the ideas in the construction of the noetherian theory and going in the same direction, we plan to carry out studies that will allow us to construct said theory for an integro-differential operator defined by an integral equation of the third type with the main part of *L* shape, and it is for this purpose that we conduct such research as a preliminary step. Various results emanating from numerous investigations devoted to integral equations of the third type are well known and can be found easily with all the necessary details in the previous cited references upstairs with some specific approaches presented. It should also be emphasized that there is a wide range of literature on specific research work undertaken on various methods of solving integral equations of which several approaches and methodologies are illustrated for example in article [29] and, relying on integral transforms related to differential operators having singularities inside the interval forming the content of the research carried out in [30].

It is very interesting to note that in order to completely cover the topic on the construction of the noetherian theory of an integro-differential operator *A*, we go further to perform the finite-dimensional extension of the initial noetherian operator *A* by adding to its initial functional space $C_{-1}^{1}[-1,1]$, successively, $D_n = \left\{\sum_{k=0}^{n} \alpha_k \delta^{\{k\}}(x)\right\}$ the space of finite linear combinations of the Delta-Dirac distribution as well as its successive Taylor derivatives on one side. On the other hand, to the same starting functional space $C_{-1}^{1}[-1,1]$, we can add $V_m = \left\{\sum_{j=1}^{m} \beta_j F.p\frac{1}{x^j}\right\}$, the functional space of improper integrals in the Hadamard sense to finally close the whole by the direct sum

$$T_{mn} = C_{-1}^{1}[-1,1] \oplus \left\{\sum_{k=0}^{n} \alpha_k \delta^{\{k\}}(x)\right\} \oplus \left\{\sum_{j=1}^{m} \beta_j F.p\frac{1}{x^j}\right\}$$

of the two spaces mentioned to the initial space. A perfect illustration of the above achievements is the subject of the content of the research conducted in our previous researches, highlighting all the detail of the principle of the conservation of the noetherity of a Noetherian operator *A* after the extension of its starting space by those mentioned above.

In the present work that we carry out in the space of continuous functions, we focus on a differential operator *L* as the main part of the integro-differential operator *A*, which is defined by an integral equation of the third type. We focus our attention only on the investigation of the operator *L* for which we carry out the construction of the Noetherian theory as it was done previously similarly in the work [31], clearly completed by the case analyzed in the article [32], with the remarkable particularity that in this indicated case, the order of the differential equation is higher and the study interval considered is closed [-1,1] rather than that [0,1] with the interior singular point zero in this situation on the middle of the interval considered. This being said and to know here, we consider for the study the linear differential operator:

$$Ly(x) = x^p y^{(n)}(x) + \gamma y^{(n-1)}(x) = f(x);\ n \geq 2, x \in [-1,1] \tag{1}$$

where $\gamma \in \mathbb{R},\ p \in \mathbb{N}$ with $f(x) \in C_0^{\{p\}}[-1,1]$ and $y(x) \in C^n[-1,1]$.

The establishment of the noetherity property (noetherian theory) of the operator L is carried out systematically through the study of the solvability of the linear differential equation of higher order n, defined by the formula (1). The application of the Dirichlet formula makes it possible to obtain the analytical formula of the expression of the solution to the equation studied and, from there, to formalize the results of the noetherization of the differential operator. The results obtained are linked to the construction of the continuity of the regularizers, as illustrated in our previous research when we first considered the interval [0,1], of which the rest of the study was extended until the set of the closed interval [-1,1]. All this allows us to calculate and determine the deficient numbers $\alpha(L)$ and $\beta(L)$ as well as the index $\chi(L)$ of the operator L, depending on the different situations analyzed in relation to the sign and the values of the parameter γ.

The content of this work is organized as follows: first and foremost, we make a detailed presentation in section 2 of the preliminaries related to the concept and notions of the well-known noetherian theory. Section 3 presents the main important results, formulated through different theorems, obtained from this work and is clearly dedicated to the cases studied, with dependence on the sign and values of γ, in other words (related to the cases a) $\gamma > 0$; b) $\gamma < 0, \gamma \neq -1$; and c) $\gamma = 0$). And to close this work, we finally summarize the content of the investigations in section 4 entitled conclusion, followed by some recommendations for the continuation of future scientific work to be undertaken.

II. PRELIMINARIES

Before presenting in full detail our main results, the following definitions and concepts well known from the noetherian theory of operators and used in our previous research are required for the realization of this study. We also recall the notions of Taylor derivatives and linear Fredholm integral equation of the third kind, widely studied in many scientific books and works such as in the references [1], [2], [3], [4], [5], [6], [7]. From papers [8], [9], [10], [11] it is presented the application of this specific notion of derivatives to reach noetherian theory. Taylor derivatives illustrated in the work [12], [13], [14], [15], [16] made it possible to guarantee the noetherian property of the investigated operator through the integral equation studied. Research carried out on the basis of the results from the books [17], [18] conduct many scientists to obtain serious results as well as illustrated in the papers [19], [20] and [21]. These authors through their researches called upon this notion which, with the same objectives, made it possible to construct the noetherian theory carried out on the basis of the important theoretical aspect from the book [22], and also when realizing out the finite-dimensional extensions of the initial noetherian operator. On this idea see also [23] and [24] in which one can find more details.
First of all, let us move to the following concept.

A) Noetherian operator.

Definition 1. Let X, Y be Banach spaces, $A \in l(X,Y)$ a linear operator. The quotient space $coker\ A = Y/imA$ is called the cokernel of the operator A. The dimensions $\alpha(A) = \dim kerA, \beta(A) = \dim cokerA$ are called the nullity and the deficiency of the operator A, respectively. If at least one of the numbers $\alpha(A)$ or $\beta(A)$ is finite, then the difference $Ind\ A = \alpha(A) - \beta(A)$ is called the index of the operator A.

Definition 2. Let X, Y be Banach spaces, $A \in l(X,Y)$ is said to be normally solvable if it possesses the following - property: The equation $Ax = y\ (y \in Y)(y \in Y)$ has at least one solution $x \in D(A)$ ($D(A)$ is the domain of A) if and only if $< y, f >= 0\ \forall f \in (im\ A)^\perp$ holds.

We recall that by the definition of the adjunct operator $(im\ A)^\perp = \ker A^*$ and it can be easily proving that the operator A is normally solvable if and only if its image space imA is closed.

Definition 3. A closed normally solvable operator A is called a noetherian operator if its index is finite.

By the way, we briefly review this important notions of Taylor derivatives which is widely used when constructing noetherian theory of the considered operator A.

Definition4. A Continuous function $\varphi(x) \epsilon C[-1,1]$ admits at the point $x = 0$ Taylor derivative up to the order $p \in \mathbb{N}$ if there exists recurrently for $k = 1,2,\dots,p$,

the following limits:

$$\varphi^{\{k\}}(0) = k! \lim_{x \to 0} x^{-k} \left[\varphi(x) - \sum_{j=0}^{k-1} \frac{\varphi^{\{j\}}(0)}{j!} x^j\right]. \tag{2}$$

The class of such functions $\varphi(x)$ is denoted $C_0^{\{p\}}[-1,1]$.

Next, let us move to the following part.

Let $C^m[-1,1], m \in \mathbb{Z}_+$, noted the Banach space of continuous functions on $[-1,1]$, having continuous derivatives up to order m, for which the norm is defined as follows:

$$\|\varphi(x)\|_{C^m[-1,1]} = \sum_{j=0}^{m} \max_{-1\le x\le 1} \left|\varphi^{(j)}(x)\right| \quad (3)$$

So, that we can consider $\varphi^{\{k\}}(0)$ are defined for all $k = 1,2,\dots,p$.

We define $C_0^{\{p\}}[-1,1]$ as a subspace of continuous functions, having finite Taylor derivatives up to order $p \in \mathbb{Z}_+$; and when $p = 0$, we put $(C_0^{\{p\}}[-1,1] = C_0^{\{0\}}[-1,1] = C[-1,1])$

Let us also define a linear operator N^k on the space $C_0^{\{p\}}[-1,1]$ by the formula:

$$(N^k\varphi)(x) = \frac{\varphi(x) - \sum_{j=0}^{k-1}\frac{\varphi^{\{j\}}(0)}{j!}x^j}{x^k}, k = 1,2,\dots,p. \quad (4)$$

One can easily verify the property $N^k = N^{k_1}N^{k-k_1}, 0 \le k_1 \le k, k, k_1 \in \mathbb{Z}_+$, where we put $N^0 = I$.

Definition 5. The operator N^p is called the characteristical operator of the space $C_0^{\{p\}}[-1,1]$.

Remark: The meaning of the previous definition can be seen from the verification of these following important lemmas below and also connected with full construction and description of the special space $C_0^{n,\{p+n-1\}}[-1,1]$.

Lemma 2.1. A function $\varphi(x)$ belongs to $C_0^{\{p\}}[-1,1]$ if and only if, the following representation

$$\varphi(x) = x^p\phi(x) + \sum_{k=0}^{p-1}\alpha_k x^k \quad (5)$$

holds with the function $\phi(x) \in C[-1,1]$ and α_k being constants.

To prove Lemma 2.1 it is enough to observe that (5) implies that the Taylor derivatives of $\varphi(x)$ up to the order p exists, and more $\varphi^{\{k\}}(0) = k!\,\alpha_k, k = 0,1,2,\dots,p-1, \varphi^{\{0\}}(0) = p!\,\phi(0)$

with $\phi(x) = (N^k\varphi)(x)$. Conversely, if $\varphi(x)$ belongs to $C_0^{\{p\}}[-1,1]$, and we define $\phi(x) = (N^k\varphi)(x)$ with $\alpha_k = \frac{\varphi^{\{k\}}(0)}{k!}, k = 0,1,2,\dots,p-1$, then the representation (5) holds. From Lemma 2.1, it follows that for $\varphi(x) \in C_0^{\{p\}}[-1,1]$ the inequality

$$\varphi(x) = x^p(N^k\varphi)(x) + \sum_{k=0}^{p-1}\frac{\varphi^{\{k\}}(0)}{k!}x^k, \quad (6)$$

is valid.

Consequently, the linear operator N^p establishes a relation between the spaces $C_0^{\{p\}}[-1,1]$ and $C[-1,1]$. The space $C_0^{\{p\}}[-1,1]$ with the norm

$$\|\varphi\|_{C_0^{\{p\}}[-1,1]} = \|N^p\varphi\|_{C[-1,1]} + \sum_{k=0}^{p-1}\left|\varphi^{\{k\}}(0)\right| \quad (7)$$

becomes a Banach space one.

Let note also that we can define the previous norm in the following way:

$$\|\varphi\|_{C_0^{\{p\}}[-1,1]} = \|N^p\varphi\|_{C[-1,1]} + \sum_{k=0}^{p-1}|\alpha_k| = \|\phi(x)\|_{C[-1,1]} + \sum_{k=0}^{p-1}|\alpha_k|$$

.

Sometimes it is comfortable and suitable to consider as norm in the space $C_0^{\{p\}}[-1,1]$ the equivalent norm defined as follows:

$$\|\varphi\|_{C_0^{\{p\}}[-1,1]} = \sum_{j=0}^{p}\left\|N^j\varphi\right\|_{C[-1,1]}$$

We can also note a very useful and clearly helpful next inequality:

$$\|\varphi\|_{C[-1,1]} \le \|N^p\varphi\|_{C[-1,1]} + \sum_{j=0}^{p-1}\left|\varphi^{\{j\}}(0)\right| = \|\varphi\|_{C_0^{\{p\}}[-1,1]}$$

Therefore, it is obvious to see that $\|\varphi\|_{C[-1,1]} \le \|\varphi\|_{C_0^{\{p\}}[-1,1]}$.

Finally, note that from the Lemma 2.1 it follows the following fact, if $\varphi(x) \in C[-1,1]$, then $x^p\varphi(x) \in C_0^{\{p\}}[-1,1]$. This assertion may be generalized as follows.

Lemma 2.2. Let $p \in \mathbb{N}, s \in \mathbb{Z}_+$. If $\varphi(x) \in C_0^{\{s\}}[-1,1]$ then, $x^p\varphi(x) \in C_0^{\{p+s\}}[-1,1]$, and the formula holds

$$\left(x^p\varphi(x)\right)^{\{j\}}(0) = \begin{cases} 0, \; j = 0,1,\dots,p-1, \\ \frac{j!}{(j-p)!}\varphi^{\{j-p\}}(0), j = p,\dots,p+s. \end{cases} \quad (8)$$

Proof. Note that a stronger assumption on the function. $\varphi(x)$, such that $\varphi(x) \in C_0^{\{p+s\}}[-1,1]$ would allow us to easily prove the lemma just by applying Leibniz formula.

For $s = 0$ the statement has already been proved above, so $x^p\varphi(x) \in C_0^{\{p\}}[-1,1]$, and $\left(x^p\varphi(x)\right)^{\{j\}}(0) = 0, j = 0,\dots,p-1$ and $\left(x^p\varphi(x)\right)^{\{p\}}(0) = p!\,\varphi(0)$. Now let us prove that $\left(x^p\varphi(x)\right)^{\{j\}}(0) = \frac{j!}{(j-p)!}\varphi^{\{j-p\}}(0), j = p+1,\dots,p+s$.

Since the derivatives are defined recurrently, and (8) is true for $j = p$, then it is sufficient to verify the passage from j to $j+1$.

We have:

$$\left(x^p\varphi(x)\right)^{\{j+1\}}(0) = (j+1)!\lim_{x\to 0}\frac{x^p\varphi(x) - \sum_{l=p}^{j}\frac{x^l}{(l-p)!}\varphi^{\{l-p\}}(0)}{x^{j+1}} \quad (9)$$

$$= (j+1)!\lim_{x\to 0}\frac{\varphi(x) - \sum_{l=0}^{j-p}\frac{x^l\varphi^{\{l\}}(0)}{l!}}{x^{j+1-p}} = \frac{(j+1)!}{(j+1-p)!}\varphi^{\{j+1-p\}}(0) \quad (10)$$

Lemmas 2.1 and 2.2 imply the next important lemma.

Lemma2.3. Let $f(x) \in C_0^{\{p\}}[-1,1], p \in \mathbb{N}$ and $f(0) = \dots = f^{\{r-1\}}(0) = 0, 1 \le r \le p$.

Then $\frac{f(x)}{x^r} \in C_0^{\{p-s\}}[-1,1]$.

B) Associated operator and associated space.

Definition 6. The Banach space $E' \subset E^*$ is called associated space with the space E, if $|(f,\varphi)| \le c\|f\|_{E'}\|\varphi\|_E$ for every $\varphi \in E, f \in E'$.

We note that the initial space E can be considered associated with the space E'. Moreover, the norm $\|f\|_{E'}$ is not obliged to be equivalent to the norm $\|f\|_{E^*}$.

Let be noted $\mathcal{L}(E_1,E_2)$ the banach algebra of all linear bounded operators from E_1 into E_2.

Definition 7. Let $E_j, j = 1,2$ two banach spaces and E'_j their associated spaces. The operators $A \in \mathcal{L}(E_1,E_2)$ and $A' \in \mathcal{L}(E'_2,E'_1)$ are called associated, if $(A'f,\varphi) = (f,A\varphi)$ for all $f \in E'_2$ and $\varphi \in E_1$.

It seems that, it is possible to formalize the noetherian property in terms of associated operator and associated space following these lemmas.

Lemma 2.4 Let $E_j, j = 1,2$ two banach spaces and E'_j their associated spaces and, let $A \in \mathcal{L}(E_1,E_2)$ and $A' \in \mathcal{L}(E'_2,E'_1)$ be associated noetherian operators and more,

$$\alpha(A) = -\alpha(A').$$

Then, for the solvability of the equation $A\varphi = f$ it is necessary and sufficient that $(f,\psi) = 0$ for all solutions of the homogeneous associated equation $A'\psi = 0$.

Next, let us move to the presentation of the general important results of the work in the following section.

III. MAIN RESULTS

In this section we undertake properly the noetherization theory for the investigation of the operator L.

Namely, here we consider as a model to be investigated, the higher order linear differential equation defined by the formula (1).

It is clear that the results on such equation could be receive from those obtained in the situation $n = 1$, making the change of function $z(x) = y^{(n-1)}(x)$ transforms the equation (1) into the following equation

$$x^p z'(x) + \gamma z(x) = f(x) \quad (11)$$

which, we already investigated in full detail in [32].

As previously, we consider $f(x) \in C_0^{\{p\}}[-1,1]$. Note that the results of the investigation of such equation which, we already investigated with full details in [32].

Therefore, let us note that the demonstrations of all the theorems formulated below are obtained in the same way as those carried out in relation to the different simple cases, meticulously analyzed and exposed when $n = 1$. Consequently, it would be judicious and useful to refer to it to similarly obtain the proofs of said theorems in the new cases mentioned in the present work.

We begin the analysis of the equation (1) with the case $p = 1$.

A. The Euler case situation.

Let the operator L defined by the formula (1) and $p = 1$ i.e $Ly(x) = x^p y^{(n)}(x) + \gamma y^{(n-1)}(x) = f(x);\ n \geq 2, x \in [-1,1]$

A. The case $\boldsymbol{\gamma > 0}$.

From the paper [32] when $p = 1$ in the space $C^1[-1,1]$, we obtained the form of the solution by the following way :

$$y^{(n-1)}(x) = z(x) = \begin{cases} \int_0^x \left(\frac{t}{x}\right)^\gamma (Nf)(t)dt + \frac{f(0)}{\gamma},\ x > 0, \\ \frac{f(0)}{\gamma}, \qquad x = 0 \\ -\int_x^0 \left(\frac{|t|}{|x|}\right)^\gamma (Nf)(t)dt + \frac{f(0)}{\gamma},\ x < 0. \end{cases} \tag{12}$$

with respect to the specific way of the formula (12) the solution of the equation $y'(x) = g(x)$ on the segment $[-1,1]$ we will take by the formula

$$y(x) = \begin{cases} \int_0^x g(t)dt\ , x \geq 0, \\ -\int_x^0 g(t)dt, x \leq 0. \end{cases} \tag{13}$$

In the case of the equation of higher order than one in (13) then we obtain multiple integration, to which (separately by $x > 0$ and by $x < 0$) the Dirichlet formula is applicable. All that after some small computations leads us to the following:

$$y(x) = P_{n-2}(x) + \frac{f(0)}{\gamma(n-1)!}x^{n-1} + \begin{cases} \int_0^x k(x,s)(Nf)(s)ds\ , x \geq 0, \\ \int_x^0 k(x,s)(Nf)(s)ds,\ x < 0, \end{cases} \tag{14}$$

where the kernel $k(x,s)$ in (14) is defined by the following formula:

$$k(x,s) = \begin{cases} \int_s^x \frac{(x-t)^{n-2}}{(n-2)!}\left(\frac{s}{t}\right)^\gamma dt,\ x > 0, s > 0, \\ \frac{(-1)^n}{(n-2)!}\int_x^s \left(\frac{|s|}{|t|}\right)^\gamma (t-x)^{n-2}dt,\ x < 0, s < 0. \end{cases} \tag{15}$$

Therefore, it holds the following theorem.

Theorem 1:

Let $\gamma > 0$, and $f(x) \in C_0^{\{1\}}[-1,1]$. The equation (11) is solvable in the space $C^n[-1,1]$ and has the solution of the form (14). The corresponding operator $L: C^n[-1,1] \to C_0^{\{1\}}[-1,1]$ is a noetherian operator with the characteristic numbers of the form $(n-1,0)$ and the index $\chi(L) = n - 1$.

Proof. Similarly as in the simple case when $n = 1$ completely investigated in [32]. In other words, first give the exact operational interpretation by introducing the inverse operator L^{-1} where L is defined by (1). So that let $L^{-1} : C_0^{\{1\}}[-1,1] \to C^n[-1,1]$ and L^{-1} defined by the obtained following formula

$$(L^{-1}f)(x) = P_{n-2}(x) + \frac{f(0)}{\gamma(n-1)!}x^{n-1} + \begin{cases} \int_0^x k(x,s)(Nf)(s)ds\ , x \geq 0, \\ \int_x^0 k(x,s)(Nf)(s)ds, \qquad x < 0, \end{cases}$$

Of course it is simply sufficient to take the expression of the solution defined by (14) as $(L^{-1}f)(x)$ analogous to that of the case of $n = 1$ expressed by (17) defining the operator L^{-1} similar to our formula (14), with the kernel designated by the term (15). It is then necessary to obtain by the same scheme, the result on the fact that the two operators L and L^{-1} are bounded, by relying on Lemma 3.1 demonstrated in [32]. From this, it follows that both operators $L: C^n[-1,1] \to C_0^{\{1\}}[-1,1]$ and $L^{-1} : C_0^{\{1\}}[-1,1] \to C^n[-1,1]$ are bounded and invertible. Next, we can see easily the relations $LL^{-1}f = f, f \in C_0^{\{1\}}[-1,1]$ and $(L^{-1}Ly)(x) = y(x)$ take place as proven in the formulas (27) and (28).

b) The case $\boldsymbol{\gamma < 0, \gamma \neq -1}$.

Under investigation of equation (11) similar to the previous case investigated when $n = 1$ in the paper [32], we obtain the following relationship:

$$y^{(n-1)}(x) = z(x) = c_1|x|^{-\gamma} + c_2|x|^{-\gamma} signx + \frac{f(0)}{\gamma}(1-|x|^{-\gamma}) + \begin{cases} -\int_x^1 \left(\frac{t}{x}\right)^{\gamma} (Nf)(t)dt, x \geq 0, \\ \int_{-1}^x \left(\frac{|t|}{|x|}\right)^{\gamma} (Nf)(t)dt, x < 0. \end{cases} \quad (16)$$

and some computations similarly as those realized when we investigated the case $\gamma > 0$ give us the description of the solution in the following way:

$$y(x) = P_{n-2}(x) + \frac{f(0)}{\gamma(n-1)!}x^{n-1} + \begin{cases} \int_0^1 k(x,s)(Nf)(s)ds + \frac{-f(0)}{\gamma(n-2)!}\int_0^x (x-t)^{n-2}(-t)^{-\gamma}dt \\ \int_{-1}^0 k(x,s)(Nf)(s)ds + \frac{(-1)^n f(0)}{\gamma(n-2)!}\int_x^0 (x-t)^{n-2}|t|^{-\gamma}dt \end{cases} \begin{cases} \frac{c_1}{(n-2)!}\int_0^x (x-t)^{n-2}t^{-\gamma}dt, x \geq 0 \\ \frac{(-1)^n c_2}{(n-2)!}\int_x^0 (t-x)^{n-2}|t|^{-\gamma}dt, x < 0, \end{cases} \quad (17)$$

where the kernel $k(x,s)$ has the following form:

$$k(x,s) = \begin{cases} \frac{-1}{(n-2)!}\int_0^{\min(x,s)} \left(\frac{s}{t}\right)^{\gamma} (x-t)^{n-2}\, dt,\ x > 0, s > 0, \\ \frac{(-1)^{n-1}}{(n-2)!}\int_{\max(x,s)}^0 \left(\frac{|s|}{|t|}\right)^{\gamma} (x-t)^{n-2}\, dt,\ x < 0, s < 0. \end{cases} \quad (18)$$

Therefore, it holds the following theorem:

Theorem 2: Let $\gamma < -1$, and $f(x) \in C_0^{\{1\}}[-1,1]$. The equation (11) is solvable in the space $C^n[-1,1]$ and has the solution of the form (17) where the kernel $k(x,s)$ is defined by the formula (18) with c_1 and c_2 – arbitrary constants. The corresponding operator $L: C^n[-1,1] \to C_0^{\{1\}}[-1,1]$ is a noetherian operator with the characteristic numbers of the form $(n+1,0)$ and the index $\chi(L) = n+1$.

Proof. Similarly, as in the situation of the demonstration of the previous theorem, we arrive by the same considerations to prove that the two operators $L: C^n[-1,1] \to C_0^{\{1\}}[-1,1]$ defined by (1) and

$$y(x) = P_{n-2}(x) + \frac{f(0)}{\gamma(n-1)!}x^{n-1} + \begin{cases} \int_0^1 k(x,s)(Nf)(s)ds + \frac{-f(0)}{\gamma(n-2)!}\int_0^x (x-t)^{n-2}(-t)^{-\gamma}dt \\ \int_{-1}^0 k(x,s)(Nf)(s)ds + \frac{(-1)^n f(0)}{\gamma(n-2)!}\int_x^0 (x-t)^{n-2}|t|^{-\gamma}dt \end{cases} + \begin{cases} \frac{c_1}{(n-2)!}\int_0^x (x-t)^{n-2}t^{-\gamma}dt, x \geq 0 \\ \frac{(-1)^n c_2}{(n-2)!}\int_x^0 (t-x)^{n-2}|t|^{-\gamma}dt, x < 0, \end{cases}$$

with the kernel $k(x,s)$ is defined by (18) are bounded and invertible in the functional spaces relating to them.

c) The case $\gamma = 0$.

This case is trivial and we can easily state the following theorem:

Theorem 3: Let $\gamma = 0$, and $f(x) \in C_0^{\{p\}}[-1,1]$. For the solvability of the equation (11) in the space $C^n[-1,1]$, it is necessary and sufficient the accomplishment of the conditions

$$f(0) = f^{\{1\}}(0) = \cdots = f^{\{p-1\}}(0) = 0.$$

Under realization of these conditions, the solution of the equation (11) is given by the formula of the following form:

$$y(x) = P_{n-2}(x) + \frac{1}{(n-2)!}\int_{-1}^x (x-t)^{n-2}(N^p f)(t)dt \quad (19)$$

where $P_{n-2}(x)$ is an $(n-1)th$ – order polynomial with arbitrary coefficients. The corresponding operator $L: C^n[-1,1] \to C_0^{\{p\}}[-1,1]$ is a noetherian operator with the characteristic numbers of the form $(n-1,p)$ and the index $\chi(L) = n-1-p$.

Proof. The proof as in the case of the similar situation studied when $n = 1$ is based on the simple decomposition of the operators L and L^{-1} into the sum of two operators in the following way $(LL^{-1}f)(x) = (I-P_1)f$, where $(P_1 f)(x) = f(0)$ and $(L^{-1}Ly)(x) = (I-P_2)y$ with $(P_2 y)(x) = y(-1)$ from where P_1 and P_2 are the projection operators, *I*- the unitary operator.

Non Euler case (case $p \geq 2$*)*

Let the operator $L: C_0^{n,\{p+n-1\}}[-1,1] \to C_0^{\{p\}}[-1,1]$ given by the left-hand side of the equality (11) be defined onto the functions from the space $C_0^{n,\{p+n-1\}}[-1,1]$.

a) The case $\gamma > 0$.

In this case we have the following result easily deduced when making the substitution.

$$y^{(n-1)}(x) = z(x) = ce_{-}^{\frac{\gamma}{p-1}x^{1-p}} + \begin{cases} e^{\frac{\gamma}{p-1}x^{1-p}} \int_0^x e^{-\frac{\gamma}{p-1}t^{1-p}} f(t)\frac{dt}{t^p}, \ x > 0, \\ \frac{f(0)}{\gamma}, \quad x = 0 \\ e^{\frac{\gamma}{p-1}x^{1-p}} \int_{-1}^x e^{-\frac{\gamma}{p-1}t^{1-p}} f(t)\frac{dt}{t^p}, \ x < 0. \end{cases} \tag{20}$$

where c is an arbitrary constant. Taking into account the formula (13) and applying the Dirichlet formula, we obtain the following result:

$$y(x) = P_{n-2}(x) + \frac{(-1)^{n-1}c}{(n-2)!}\left[\int_x^0 (t-x)^{n-2} e^{\frac{\gamma}{p-1}t^{1-p}} dt\right]_{-} + \begin{cases} \int_0^x k(x,s)f(s)ds, x > 0, \\ \int_{-1}^0 k(x,s)f(s)ds, x < 0. \end{cases} \tag{21}$$

where the kernel $k(x,s)$ is defined by:

$$k(x,s) = \begin{cases} \frac{1}{(n-2)!}\int_s^x \frac{e^{-\frac{\gamma}{p-1}s^{1-p}}}{s^p}(x-t)^{n-2} e^{\frac{\gamma}{p-1}t^{1-p}} dt, \ x > 0, s > 0, \\ \frac{(-1)^{n-1}}{(n-2)!}\int_{\max(s,x)}^0 \frac{e^{-\frac{\gamma}{p-1}s^{1-p}}}{s^p}(t-x)^{n-2} e^{\frac{\gamma}{p-1}t^{1-p}} dt, x < 0, s < 0, \end{cases} \tag{22}$$

and $P_{n-2}(x)$ a polynomial with arbitrary coefficients.
So that we see it holds the following theorem:

Theorem 4: Let $\gamma > 0$, $f(x) \in C_0^{\{p\}}[-1,1], p \geq 2 \ and \ n \geq 2$. The equation (11) is solvable in the space $C_0^{n,\{p+n-1\}}[-1,1]$ with a right-hand side $f(x)$ and the solution is given by the formula (21) where the kernel $k(x,s)$ has the form (22). The corresponding operator $L: C_0^{n,\{p+n-1\}}[-1,1] \rightarrow C_0^{\{p\}}[-1,1]$ is Noetherian with the characteristic numbers $(n,0)$ and the index $\chi(L) = n$.

Proof. Obvious as in the case of the simple situation when n=1.

b) The case $\gamma < 0$.

From our previous obtained solution of the equation (1) from we have:

$$y^{(n-1)}(x) = z(x) = ce_{+}^{\frac{\gamma}{p-1}x^{1-p}} + \begin{cases} -e^{\frac{\gamma}{p-1}x^{1-p}} \int_x^1 e^{-\frac{\gamma}{p-1}t^{1-p}} f(t)\frac{dt}{t^p}, \ x > 0, \\ \frac{f(0)}{\gamma}, \quad x = 0 \\ -e^{\frac{\gamma}{p-1}x^{1-p}} \int_x^0 e^{-\frac{\gamma}{p-1}t^{1-p}} f(t)\frac{dt}{t^p}, \ x < 0. \end{cases} \tag{23}$$

where c is an arbitrary constant. From that and using the same method as upstairs we find

$$y(x) = P_{n-2}(x) + \left[\frac{c}{(n-2)!}\int_0^x (x-t)^{n-2} e^{\frac{\gamma}{p-1}t^{1-p}} dt\right]_{+} + \begin{cases} \int_0^1 k(x,s)f(s)ds, x > 0, \\ \int_x^0 k(x,s)f(s)ds, x < 0. \end{cases} \tag{24}$$

where the kernel $k(x,s)$ in the formula (24) has the following form:

$$k(x,s) = \begin{cases} -\frac{1}{(n-2)!}\int_{\min(x,s)}^0 \frac{e^{-\frac{\gamma}{p-1}s^{1-p}}}{s^p}(x-t)^{n-2} e^{\frac{\gamma}{p-1}t^{1-p}} dt, \ x > 0, s > 0, \\ \frac{(-1)^n}{(n-2)!}\int_x^s \frac{e^{-\frac{\gamma}{p-1}s^{1-p}}}{s^p}(t-x)^{n-2} e^{\frac{\gamma}{p-1}t^{1-p}} dt, \ x < 0, s < 0, \end{cases} \tag{25}$$

and $P_{n-2}(x)$ a polynomial with arbitrary coefficients.
Now let us formulate the last theorem as follows.

Theorem 5: Let $\gamma < 0$, $f(x) \in C_0^{\{p\}}[-1,1], p \geq 2 \ and \ n \geq 2$. The equation (11) is solvable in the space $C_0^{n,\{p+n-1\}}[-1,1]$ with any right-hand side $f(x)$ and the solution is given by the formula (24) where the kernel $k(x,s)$ has the form (25). The corresponding operator $L: C_0^{n,\{p+n-1\}}[-1,1] \rightarrow C_0^{\{p\}}[-1,1]$ is Noetherian with the characteristic numbers $(n,0)$ and the index $\chi(L) = n$.

Proof. Refer to the demonstration of the theorem in the completely studied analogous situation when $n = 1$, completely investigated by applying Lemma 3.2, Lemma 3.3, and Lemma 3.4 in [32].

To close our research carried out in this work, let's go straight to the next part entitled Conclusion.

IV. CONCLUSION.

This scientific work presents in detail the different stages of the complete investigation of the realization of the construction of the Noetherian theory for the differential operator *L*, defined by a linear singular differential equation of higher order *n* in the space of continuous functions $C^n[-1,1]$. We have identified the solvency conditions of equation (1) taking into account the nature of the parameter $\gamma \in \mathbb{R}$ in various analyzed situations. Considering the above, we have managed to progressively determine the deficient numbers (α, β) characterizing the operator *L*, with $\alpha(L) = \dim kerL, \beta(L) = \dim cokerL$ called the nullity and the deficiency of the operator L, respectively and, consequently, subsequently the index $Ind\ L = \chi(L) = \alpha(L) - \beta(L)$ of the operator L is defined in an induced manner, which is a finite number in all the cases studied, making the operator considered a Noetherian operator.

It is clear to mention quite naturally that in this investigation carried out, we were able to conduct the most important and necessary study to carry out the construction of noetherity of an integro-differential operator *A*, defined by a singular integral equation of the third type having for the main part, the studied operator L considered. We know from the general theory that, under perturbation of a noetherian operator by a compact operator and, in the case to be investigated in a brief future, we will reach and maintain the noetherity nature of the initial operator L. This will be the next future work to undertake when we do consider first of all, the operator *A* as a sum of two operators L and K where, L is the operator defined by $Ly(x) = x^p y^{(n)}(x) + \gamma y^{(n-1)}(x) = f(x)$ and K is a compact operator defined as follows $K\varphi = \int_{-1}^{1} k(x,t)y(t)dt.$ In other words, using the results obtained in this work relative to its principal part *L*, we will arrive at the noetherization of the operator *A* defined by the following third kind integral equation

$$Ay(x) = x^p y^{(n)}(x) + \gamma y^{(n-1)}(x) + \int_{-1}^{1} K(x,t)\,\varphi(t)dt = f(x)$$

in the functional space chosen adequately taking into account the compactness of the operator *K*.

It is worth noting that the results of the investigation of the noethery of the operator L will not only lead to the construction of the complete noetherian theory of an integro-differential operator A defined by an integral equation of the third kind in suitable functional spaces having as their principal part the said operator L, but will also bring out the conditions of solvability of the operator A in parallel.

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Contribution of individual authors to the creation of a scientific article (ghostwriting policy)

Chen Lee carried out the simulation and the optimization.
Kemal Mehmet has implemented the Algorithm 3.2 and 15.1 in Java
George Luton has organized and executed the experiments of Section 4.
Michael Walton was responsible for the Simulation and Statistics.

We confirm that all Authors equally contributed to the present research, at all stages from the formulation of the problem to the final findings and solution.

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