energy $E(G)$ of a Semigraph $G(V,X)$ of order n, size m, as

$$\sqrt{2\sum_{e\in X}\left(1^2+2^2+\ldots+k_e^{\ 2}\right)}\leq E(G)\leq\sqrt{2n\sum_{e\in X}\left(1^2+2^2+\right.}$$

Where $e\in X$ be an edge of cardinality k_e+1 .

Further, [8], in 2021, introduced the concept of *Distance matrix and energy of semigraph*.

A set of vertices that covers all the edges is a vertex cover for a semigraph *G*. And a set of vertex with minimum cardinality covering all the edges of G is called its minimum covering. The minimum cardinality of a vertex cover is called vertex covering number and it is denoted by $\alpha_0=\alpha_0(G)$.

Here a new type of matrix is introduced, called the minimum covering matrix of a Semigraph. Further studies its singular values and energy.

In, [12], the authors introduced adjacency matrix of signed semigraph and derived some properties.

II. Minimum covering matrix and its energy of a semigraph

In the year 2017, [9], defined an adjacency matrix associated with a semigraph as

The adjacency matrix of a semigraph:

Let $G(V,X)$ be a semigraph with vertex set $V=\{v_1,v_2,\ldots\ldots..,v_n\}$ and edge set $X=\{e_1,e_2,e_3,\ldots..,e_m\}$. The Adjacency matrix $G(V,X)$ is a $n\times n$ matrix $A=[a_{ij}]$ defined as follows:

i. For every edge e_i of *X* of cardinality, say *k*, let $e_i=(i_1,i_2,i_3,\ldots\ldots..,i_k)$ such that $i_1,i_2,i_3,\ldots\ldots..,i_k$ are distinct vertices in V , for all $i_r\in e_i$; $r=1,2,\ldots,k$

$(a)\ a_{i_1i_r}=r-1$

$(b)\ a_{i_ki_r}=k-r$

ii. All the remaining entries of A are zero.

Thus, we defined the minimum covering matrix of a semigraph as follows

A. The minimum covering matrix of a semigraph:

If $G(V,X)$ be a semigraph order *n* and, size *m*. Let *C* be the minimum covering set, then the minimum covering matrix of *G* is the square matrix $A_{mc}(G)=(a_{ij})$ of order n, where

i. For every edge e_i of *X* of cardinality, say *k*, let $e_i=(i_1,i_2,i_3,\ldots\ldots..,i_k)$ such that $i_1,i_2,i_3,\ldots\ldots..,i_k$ are distinct vertices in V , for all $i_r\in e_i$; $r=1,2,\ldots,k$

$(a)\ a_{i_1i_r}=r-1,$

$(b)\ a_{i_ki_r}=k-r$

ii. $a_{ij}=1$ if $i=j$ and $v_i\in C$.

iii. All the remaining entries of *A* are zero.

B. The minimum covering energy of semigraph:

In [10], the author defined the energy of a general matrix (of any size) as the summation of the singular values of that matrix.

Thus, If $\sigma_1,\ \sigma_2,\ \ldots\ldots\ldots.,\sigma_n$ be the singular values of the minimum covering matrix $A_{mc}(G)$ of the semigraph *G*, then the minimum covering energy of a semigraph denoted by $E_{mc}(G)$, is defined as the summation of its singular values. i.e.

$$E_{mc}(G)=\sum_{i=1}^{n}\sigma_i$$

We observe that, $A_{mc}(G)A'_{mc}(G)$ is a positive semidefinite matrix. So its eigenvalues $\lambda_1,\ \lambda_2,\ \ldots\ldots\ldots\ldots.,\lambda_n$ are non-negative and therefore the singular values of $A_{mc}(G)$ are non-negative real numbers. Thus $E_{mc}(G)\geq 0$, equality holds if and only if the number of edges in *G* is zero.

The minimum covering energy of a semigraph is well defined, as if G' be a semigraph obtained by relabeling of the vertices of *G*, then $A_{mc}(G')A'_{mc}(G')$ is obtained by interchanging the rows and the corresponding columns of $A_{mc}(G)A'_{mc}(G)$. Hence the eigenvalues of $A_{mc}(G)A'_{mc}(G)$ and $A_{mc}(G')A'_{mc}(G')$ are the same, and so the singular values of G and G' are also the same.

III. Properties of minimum covering energy of semigraph

Theorem 1 Let $A_{mc}(G)$ is the minimum covering matrix of a semigraph G, and C is its minimum covering set. If $\lambda_1,\ \lambda_2,\ \ldots.,\lambda_n$ are eigenvalues of $A_{mc}(G)A'_{mc}(G)$. Then

$$\sum_{i=1}^{n}\lambda_i=2\sum_{e\in X}\left(1^2+2^2+\ldots+k_e^{\ 2}\right)+|C|$$

where the cardinality of an edge $e\in X$ of the semigraph is k_e+1 and $k_e\geq 1$.

Proof: In the minimum covering matrix $A_{mc}(G)$, corresponding to every edge $e\in X$ of cardinality k_e+1 ,

there is a sequence $\{1, 2, ..., k_e\}$ in the rows corresponding to the end vertices of that edge. And there are $|C|$ nos. of 1's in the diagonal of $A_{mc}(G)$. Thus every edge contributes $2\sum_{e}\left(1^2 + 2^2 + ... + k_e^{\,2}\right)$ and the diagonal elements contribute $|C| \times 1^2$ to the trace of $A_{mc}(G)A'_{mc}(G)$.
Therefore

$$trac(A_{mc}A'_{mc}) = 2\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e^{\,2}\right) + |C| \times 1^2$$

Hence

$$\sum_{i=1}^{n}\lambda_i = 2\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e^{\,2}\right) + |C|$$

Theorem 2. The minimum covering energy $E_{mc}(G)$ of a semigraph *G*, is a square root of an even or odd integer according as $|C|$ is even or odd.

Proof: If $\sigma_1, \sigma_2, ..., \sigma_n$ be the singular values of the minimum covering matrix $A_{mc}(G)$ of the semigraph *G*, then

$$(\sigma_1 + \sigma_2 + ... + \sigma_n)^2 = \sum_{i=1}^{n}\sigma_i^2 + 2\sum_{i<j}\sigma_i\sigma_j$$

Thus

$$[E_{mc}(G)]^2 = \sum_{i=1}^{n}\lambda_i + 2\sum_{i<j}\sigma_i\sigma_j$$

$$= 2\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e^{\,2}\right) + |C| + 2\sum_{i<j}\sigma$$

$$= 2\left[\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e^{\,2}\right) + \sum_{i<j}\sigma_i\sigma_j\right]$$

$$E_{mc}(G) = \sqrt{2\left[\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e^{\,2}\right) + \sum_{i<j}\sigma_i\sigma_j\right] + |C}$$

Thus the minimum covering energy $E_{mc}(G)$ of a semigraph *G*, is a square root of an even or odd integer according as $|C|$ is even or odd.

Theorem 3. The minimum covering energy $E_{mc}(G)$ of a semigraph *G*, then

$$[E_{mc}(G)]^2 = |C| (\bmod 2)$$

Proof: By Theorem 2**,** the minimum covering energy $E_{mc}(G)$ of a semigraph *G*, is a square root of an even or odd integer according to $|C|$ is even or odd. i.e.

$$E_{mc}(G) = \sqrt{2t + |C|}$$ where t is a positive integer.

$$[E_{mc}(G)]^2 = 2t + |C|$$

Thus

$$[E_{mc}(G)]^2 = |C| (\bmod 2)$$

IV. SOME BOUNDS ON MINIMUM COVERING ENERGY OF SEMIGRAPH

Theorem 4. For a semigraph *G* on *n* vertices and *m* edges,

$$\sqrt{2\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e^2\right) + |C|} \le E_{mc}(G) \le \sqrt{n\left[2\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e^2\right) + |C|\right]}$$

Proof: Let σ_i, $i = 1, 2, ..., n$ be the singular values of A_{mc}, and λ_i, $i = 1, 2, ..., n$ be the eigenvalues of $A_{mc}(G)A'_{mc}(G)$. By Cauchy Schwarz's inequality on two vectors $(\sigma_1, \sigma_2, ..., \sigma_n)$ and $(1, 1, ..., 1)$, we have

$$(\sigma_1 + \sigma_2 + ... + \sigma_n)^2 \le n\sum_{i=1}^{n}\sigma_i^2 = n\sum_{i=1}^{n}\lambda_i$$

Thus,

$$[E_{mc}(G)]^2 \le n\left[2\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e^{\,2}\right) + |C|\right]$$

Again we have

$$[E_{mc}(G)]^2 = \left(\sum_{i=1}^{n}\sigma_i\right)^2 \ge \sum_{i=1}^{n}\sigma_i^2 = \sum_{i=1}^{n}\lambda_i$$

Thus,

$$[E_{mc}(G)]^2 \ge 2\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e^{\,2}\right) + |C|$$

Hence

$$\sqrt{2\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e^2\right) + |C|} \le E_{mc}(G) \le \sqrt{n\left[2\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e^2\right) + |C|\right]}$$

Theorem 5. If *G* is a semigraph having *n* vertices and *m* edges, then

$$[E_{mc}(G)]^2 \ge 2\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e^{\,2}\right) + |C| + n(n-1)\Delta^{1/n}$$

,

Where $\Delta = \det(A_{mc}A'_{mc})$.

Proof: Let σ_i, $i = 1, 2, ..., n$ be the singular values of $A_{mc}(G)$, then we have

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$$[E_{mc}(G)]^2 = \left(\sum_{i=1}^{n}\sigma_i\right)^2 = \sum_{i=1}^{n}\sigma_i^2 + 2\sum_{i<j}\sigma_i\sigma_j = \sum_{i=1}^{n}\lambda_i +$$

As σ_i, $i = 1, 2, ..., n$ are non-negative, so $n(n-1)$ nos. of $\sigma_i\sigma_j$ are also non-negative number.

Therefore, applying $AM \geq GM$ on $n(n-1)$ nos. of non-negative numbers $\sigma_i\sigma_j$. We have

$$\frac{1}{n(n-1)}\sum_{i\neq j}\sigma_i\sigma_j \geq \left(\prod_{i\neq j}\sigma_i\sigma_j\right)^{\frac{1}{n(n-1)}} = \left(\prod_{i=1}^{n}\sigma_i^{2(n-1)}\right)^{\frac{1}{n(n}}$$

i.e.

$$\sum_{i\neq j}\sigma_i\sigma_j \geq n(n-1)\left(\prod_{i=1}^{n}\lambda_i^{n-1}\right)^{\frac{1}{n(n-1)}} = n(n-1)\left(\prod_{i=1}^{n}\lambda_i\right)^{\frac{1}{n}}$$

Thus $$\sum_{i\neq j}\sigma_i\sigma_j \geq n(n-1)\Delta^{\frac{1}{n}}$$

Where $\Delta = \prod_{i=1}^{n}\lambda_i = \det(A_{mc}A'_{mc})$

Therefore we get

$$[E_{mc}(G)]^2 \geq \sum_{i=1}^{n}\lambda_i + n(n-1)\Delta^{\frac{1}{n}}$$

By Theorem 1 we obtain

$$[E_{mc}(G)]^2 \geq 2\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e{}^2\right) + |C| + n(n-1)\Delta^{\frac{1}{n}}$$

Lemma:1, [10]. If $A = [a_{ij}]$ is any non-constant matrix and its norm defined as $\| A \|_2 = \sqrt{\sum_{ij} a_{ij}^2}$. Suppose $\sigma_1 \geq \sigma_2 \geq .... \geq \sigma_n$ are singular values of *A*, then

$$E(A) \geq \sigma_1 + \frac{\| A \|_2^2 - \sigma_1^2}{\sigma_2} .$$

Thus, we evaluate a lower bound for $E_{mc}(G)$

Theorem 6. For a semigraph *G* on *n* vertices, if σ_1 and σ_2 are respectively largest and second largest singular values of its minimum covering matrix $A_{mc}(G)$. Then we have

$$E_{mc}(G) \geq \sigma_1 + \frac{2\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e{}^2\right) + |C| - \sigma_1^2}{\sigma_2}$$

Proof: By **Lemma 1**, for the minimum covering matrix $A_{mc}(G)$ of *G*, we have

$$E_{mc}(G) \geq \sigma_1 + \frac{\| A_{mc} \|_2^2 - \sigma_1^2}{\sigma_2}$$

Clearly, from the definition of the norm of a matrix we have

$$\| A_{mc}(G) \|_2^2 = trace(A_{mc}(G)A'_{mc}(G))$$

$$= 2\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e{}^2\right) + |C|$$

Hence, we get

$$E_{mc}(G) \geq \sigma_1 + \frac{2\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e{}^2\right) + |C| - \sigma_1^2}{\sigma_2}$$

Which gives another lower bound of $E_{mc}(G)$.

V. Relation between Minimum Covering Energy and Energy of Semigraph

Theorem 7 Let $G(V, X)$ be a semigraph of order *n*, size *m* then

$$E_{mc}(G) \geq \frac{E(G)}{\sqrt{n}}$$

Where $E(G)$ is the energy of semigraph *G*.

Proof: If $G(V, X)$ be a semigraph of order *n*, size *m*, Then by Theorem 2 of [7], we have

$$\sqrt{2\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e{}^2\right)} \leq E(G) \leq \sqrt{2n\sum_{e\in X}\left(1^2 + 2^2 + ... + K_e^2\right)}$$

$$2\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e^2\right) \leq [E(G)]^2 \leq 2n\sum_{e\in X}\left(1^2 + 2^2 + ... + k^2\right)$$

Thus

$$[E(G)]^2 \leq 2n\sum_{e\in X}\left(1^2 + 2^2 + ... + k^2\right)$$

Therefore

$$\frac{[E(G)]^2}{n} \leq 2\sum_{e\in X}\left(1^2 + 2^2 + ... + k^2\right)$$

If $E_{mc}(G)$ be the minimum covering energy of a semigraph $G(V, X)$, by **Theorem 5** we get

$$[E_{mc}(G)]^2 \geq 2\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e{}^2\right) + |C| + n(n-1)\Delta^{1/n}$$

i.e $$[E_{mc}(G)]^2 \geq 2\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e{}^2\right)$$

Thus $$[E_{mc}(G)]^2 \geq \frac{[E(G)]^2}{n}$$

Hence $$E_{mc}(G) \geq \frac{E(G)}{\sqrt{n}}$$

Theorem 8 For a semigraph $G(V, X)$ of order *n*, size *m*, if σ_1 and σ_2 are respectively largest and second largest singular values of its minimum covering matrix $A_{mc}(G)$. Then we have

$$nE_{mc}(G) \geq \frac{[E(G)]^2 - n\sigma_1^2}{\sigma_2}$$

Where $E(G)$ is the energy of the semigraph.

Proof : If $G(V, X)$ be a semigraph of order *n*, size *m*, Then by Theorem 2 of [7].

$$\sqrt{2\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e{}^2\right)} \leq E(G) \leq \sqrt{2n\sum_{e\in X}\left(1^2 + 2^2 + ... + K_e^2\right)}$$

Thus

$$[E(G)]^2 \leq 2n\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e^2\right)$$

By Theorem 6 we have,

$$E_{mc}(G) \geq \sigma_1 + \frac{2\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e{}^2\right) + |C| - \sigma_1^2}{\sigma_2}$$

Thus

$$\sigma_2 E_{mc}(G) - \sigma_1\sigma_2 + \sigma_1^2 \geq 2\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e^2\right) + |C|$$

i.e.

$$\sigma_2 E_{mc}(G) - \sigma_1\sigma_2 + \sigma_1^2 \geq 2\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e^2\right)$$

i.e.

$$n\left(\sigma_2 E_{mc}(G) - \sigma_1\sigma_2 + \sigma_1^2\right) \geq 2n\sum_{e\in X}\left(1^2 + 2^2 + ... + k_e^2\right)$$

i.e. $$n\left(\sigma_2 E_{mc}(G) - \sigma_1\sigma_2 + \sigma_1^2\right) \geq [E(G)]^2$$

VI. CONCLUSION AND FUTURE WORK

As it is evident from recent literature, the field of graph energy is a relatively new and rapidly developed area of research that studies various aspects of energy associated with many graph-based models in chemical graph theory. Thus, the study of the energy in the Semigraph model is a very fascinating and challenging area of research that holds great promise for addressing a wide range of real-world problems in the near future.

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