On Classical and Distributional Solutions of a Higher Order Singular Linear Differential Equation in the Space K’

Abdourahman Haman Adji[1*], Shankishvili Lamara Dmitrievna[2]
[1]Department of Mathematics, Higher Teachers’ Training College,
University of Maroua, Maroua,
Cameroon
[2]Department of Mathematics,
Georgian Technical University,
Kostova Street, Tbilisi, Georgia
[*]Corresponding author

Received: June 2, 2022. Revised: February 4, 2023. Accepted: March 22, 2023. Published: April 11, 2023.

Abstract—In this research work, we aim to find and describe all the classical solutions of the homogeneous linear singular differential equation of order *l* in the space of *K′* distributions. Recall that in our previous research, the results of which have been published in some journals, we had undertaken similar studies in the case of a singular differential equation of the Euler type of second order, when the conditions $m = r + 2,\ n = r + 1$ were carried out. That said, our intentions in this article are therefore to generalize the results obtained and recently published, focusing our research on the situation of the homogeneous singular linear differential equation of order *l* of Euler type. In this orientation, we base ourselves on the classical theory of ordinary linear differential equations and look for the particular solution to the equation considered in the form of the distribution $y(x) = x^{\lambda}$, with a parameter λ to be determined, which we replace in the latter. Depending on the nature of the roots of the characteristic polynomial of the homogeneous equation we identify, case by case, all the solutions indicated in the sense of distributions in the space *K′*. In this same work, we return to the non-homogeneous equation of order *l* of the same Euler type, whose second member consists only of the derivative of order *s* of the Dirac-delta distribution studied in our previous work, to fully describe all the solutions of the latter in the sense of distributions in the space *K′*.

We finalize this work by making an important remark emphasizing the interest in undertaking research of the same objective of finding a general solution, by studying the singular differential equations of the same higher-order *l* with the particularity of being of Euler types on the left and Euler on the right in the space of distributions *K’*.

Keywords—classical solutions, Dirac-delta function, generalized functions, test functions, zero-centered solutions.

I. INTRODUCTION

It is well known that differential equations and partial differential equations as part of mathematics are widely applied in the studies of several natural physical phenomena. By this means, modeling situations to be investigated are carried out via differential equations and partial differential equations.

From the general theory, we know that ordinary differential equations (ODE) are those in which the unknown function or functions depend on only one variable. It is quite clear that calculating a derivative is easier than solving a differential equation generally. For instance, several types of differential equations, depending on their category, can be solved easily and others with more difficulties. When the unknown function contains several independent variables, then we speak of partial differential equations (PDE) and it is known that difficulties may increase, as well as calculating partial derivatives of such functions as solving partial differential equations in which they are playing a role.

Sometimes, it may be challenging to solve some type of complicated differential equation. However, solving a differential equation is always opening the way for its solution to undertake a new study once more and, then also guide us to a complete comprehension of a physical phenomenon in our daily life. One may also understand and know that it is not easy in some specific cases to solve certain kinds of differential equations, even those of the first order. Solving a differential equation is always conducting us in our mind to set a new kind of differential equation to be investigated and solved. Depending on the spaces in which the solution of differential equations is sought, we can face unwanted situations according to the form of the solutions. For example, a simple differential equation

$$Ax^p y'(x) = \delta^{(s)}(x)$$

has the following form of a distributional solution in the space *K'*

$$y(x) = \frac{(-1)^p s!}{A(s+p)!}\delta^{(s+p-1)}(x) + \sum_{k=1}^{p-1} c_{k+1}\delta^{(k-1)}(x) + c_1\theta(x) + c_0,$$

where C_k,

$k = 1, \dots\dots, p-1$; c_1 *and* c_0 – are arbitrary constants and $\theta(x)$

is the Heaviside test function.

This result has been proved by deducing it from the formula (11) in the paper, [18].

From the previous, we can see and note that the solution of such an equation is not unique and depends on $(p+1)-$arbitrary constants.

In the upstairs studied example of the simple differential equation we applied the Fourier transform to investigate the existence of zero-centered solutions moreover this method had been also used when solving some kind of differential equations in our previous works. Analogically, but this time another method was used in [19], when other scientists applied the Laplace transform to search the generalized solutions of the fourth-order Euler differential equations

$$t^4y^{(4)}(t) + t^3y'''(t) + t^2y''(t) + ty'(t) + my(t) = 0$$

where m is an integer and $t \in \mathbb{R}$. From their work, they brought out a distributional solution and a weak solution depending on the relationship between the parameter m and some constant $k \in \mathbb{N}$. For explanations and more details on this investigation refer to [4], [5], [19]. It is necessary to note that interesting scientific works, approaching problems of the same nature devoted to the differential equations of specific types in the spaces of the distributions were carried out by certain researchers, whose remarkable results are the subject of the contents of the publications, [22], [23], [24], [25], [26], [27].

Alongside many other papers scientists again, we also mentioned the brief survey of results on distributional and entire solutions of ordinary differential equations (ODE) and functional differential equations (FDE) which have been given in the work, [7]. Within this paper, a significant clear emphasis was devoted to linear equations with polynomial coefficients, with also a special approach related to the work on generalized-function solutions of integral equations. The same direction similar to this indicated research has been investigated in the paper titled «The distributional solutions of ordinary differential equation with polynomial coefficients» published in 2001. Full details in [8].

Recall first of all that it has been investigated a first-order linear singular differential equation with the goal to look at all classical solutions to its homogeneous equation, arriving to analyze them step by step in the space *K'*. Full details on such studied problem are localized in the scientific paper, titled «On generalized-function solutions of a first-order linear singular differential equation in the space *K'* via Fourier transform. » [8]. In the cited work, we completely investigated the question of the solvency of the singular linear differential equation of the first order $Ax^p y'(x) + Bx^q y(x) = \delta^{(s)}(x)$ in the space *K'* mentioning all distributional and classical solutions which naturally verified the equation. For details see also, [6], [14], [18], [20].

Analogically, to the upstairs scientific works already conducted and serving us as a fundamental basis to generalize and completely cover a such topic, we turn to the higher order linear differential equation $\sum_{i=0}^{l} a_i\, x^{k_i} y^{(i)}(x) = \delta^{(s)}(x)$ investigated on the question of the existence of zero-centered solutions in the space *K'*, in the Euler case to describe all classical solutions. Explanations and full details related to this question and the analogical problem may be found in our recent publication, [18], and also in the papers, [11], [12], [13], [15]. Based on all that has been done and said, we naturally focus our goal to set the problem of the question of the existence and analysis of all classical solutions of the considered higher order linear singular Euler differential equation in the space of generalized functions *K'*.

So here let us consider first the following higher order homogeneous linear singular Euler differential equation defined by:

$$\sum_{i=0}^{l} a_i\, x^{k_i} y^{(i)}(x) = 0, \qquad (1)$$

where the parameters k_i satisfy the conditions $k_i = k_0 + i;\ i = 0,1\dots\dots, l$ and the distributional function $y(x)$ is to be found from the space *K'*.

The content of this research is organized in the following way: Section II is devoted to well-known concepts and notions related to some important facts of the general theory of differential equations and generalized functions in the space K'. Presenting the general results of our investigations, we are given in section III the overview of the whole research conducted with the formulated theorems and a description of the general solutions of the corresponding homogeneous equation with Dirac-delta function or its derivatives of s-order in the right-hand side firstly, and lastly, it is used previous results obtained to completely describe the solutions of the non-homogeneous. In the final part of the work, we summarize and conclude our investigation in section IV dedicated to the conclusion, followed by some remarks and necessary recommendations for the follow-up or future scientific works to undertake, stated in section V.

II. PRELIMINARIES

Before proceeding to the main results of the work, the following definitions and concepts well-known from the theory of generalized functions are required to undertake the investigation. For more details, refer among many others to the following references also, [1], [2], [3], [9], [10], [19], [21].

Definition 2.1. Let *K* be the space consisting of all real-valued functions $\varphi(x)$ with continuous derivatives of all orders and compact support. The support $\varphi(x)$ is the closure of the set of all elements
$t \in \mathbb{R}$ such that $\varphi(x) \neq 0$. Then, $\varphi(x)$ is called a test function.

Definition 2.2. A distribution T is a continuous linear functional on the space *K* on the space of the real-valued functions with infinitely-differentiable and bounded support. The space of all such distributions is denoted by K'.
For every $T \in K'$ and $\varphi(x) \in K$, the value T has on $\varphi(x)$ is denoted by $(T, \varphi(x))$. Note that $(T, \varphi(x)) \in \mathbb{R}$.

Below, let us give some examples of distributions.
(a) A locally-integrable function $g(x)$ is a distribution generated by the locally-integrable function $g(x)$. Here, we define

$$(g(x), \varphi(x)) = \int g(x)\varphi(x)dx \ .$$

integration on the support Ω, and $\varphi(x) \in K$
In this case, the distribution is called regular distribution.
(b) The Dirac delta function is a distribution defined by $(\delta(x), \varphi(x)) = \varphi(0)$, and the support of $\delta(x)$ is $\{0\}$.

In this case, the distribution is called irregular distribution or singular distribution.

Definition 2.3. The sth-order derivative of a distribution T, denoted by $T^{(s)}$, is defined by $\left(T^{(s)}, \varphi(x)\right) = (-1)^s \left(T, \varphi(x)^{(s)}\right)$ for all $\varphi(x) \in K$.
Let's give an example of derivatives of the singular distribution
$T = \delta$ we have:
a) $(\delta'(x), \varphi(x)) = -(\delta(x), \varphi'(x)) = -\varphi'(0)$;

$$\left(\delta^{(s)}, \varphi(x)\right) = (-1)^s \left(\delta(x), \varphi(x)^{(s)}\right) =$$
b) $(-1)^s \varphi(0)^{(s)}$

Definition 2.4. Let $\omega(x)$ be an infinitely-differentiable function. We define the product of $\omega(x)$ any distribution T in *K'* by $(\omega(x)T, \varphi(x)) = (T, \omega(x)\varphi(x))$ for all $\varphi(x) \in K$.
Now we can move to the important part of the work stated in the following section.

III. PRINCIPAL RESULTS OF THE WORK

The following part is presenting, step by step, completely the global results of our investigation devoted to the research of classical solutions of the considered equation in the space *K'*, which takes the form numbered (2).

$$\sum_{i=0}^{l} a_i\, x^{k_i} y^{(i)}(x) = 0, \tag{2}$$

From the general theory of differential equations, it is well-known that classical (regular) solutions may be found in the following way defined by formula (3):

$$y(x) = x^{\lambda}, \tag{3}$$

where $\lambda -$ is an unknown parameter to be determined. Putting $y(x)$ and next after computing its $i\text{-}th$ derivatives replaced into (2), with respect to the conditions on k_i leads us to the following result:

$$\sum_{i=0}^{l} a_i\, x^{k_i} y^{(i)}(x) = \sum_{i=0}^{l} a_i\, x^{k_i} (x^{\lambda})^{(i)}(x) =$$
$$\sum_{i=0}^{l} a_i\, \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-i+1) = 0 \tag{4}$$

So let us denote this polynomial as follows $P_l(\lambda)$ and we call it the characteristic equation of equation (2) :

$$P_l(\lambda) = \sum_{i=0}^{l} a_i\, \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-i+1) = 0 \tag{5}$$

Remark that $P_l(\lambda)$ is an *l*-order equation relative to the parameter λ and as it is well known, this equation admits exactly *l* roots. This will lead us to distinguish different cases for our investigation depending on the multiplicity of the roots λ_i solutions $P_l(\lambda) = 0$. Therefore, now our goal is to pick from the classical solutions of equation (1) those which are locally integrable functions. In other words, we construct from the classical solutions, the corresponding solutions in the meaning of distributions from the space K'. So, let us enumerate the following situations described below and then we have:

1) All the roots λ_i of the polynomial $P_l(\lambda)$are simple different real roots,

such that $\lambda_i > -1, (i = 1, \ldots\ldots, l)$. Then, the solution of the equation in the sense of *K' is* defined by the formula :

$$y(x) = \sum_{i=1}^{l} c_i\, x^{\lambda_i} \theta(x) + \sum_{i=1}^{l} c_i{}'\, |x|^{\lambda_i} \theta(-x), \tag{6}$$

where $c_i, c_i{}' (i = 1 \ldots\ldots, l)$ are arbitrary constants and $\theta(x) -$ is the Heaviside test function.
2) Next, we consider the case when the roots of the characteristic equation $P_l(\lambda) = 0$ denoted λ_i are *m* - times roots. Then, if $\lambda_i > -1$, the solution of equation (2) corresponding to this root is defined by the formula :

$$y(x) = \sum_{j=1}^{m} c_j\, x^{\lambda_i} (lnx)^{j-1} \theta(x) + \sum_{j=1}^{m} c_j{}'\, |x|^{\lambda_i} (ln|x|)^{j-1} \theta(-x)$$

DOI: 10.46300/91019.2023.10.1 Volume 10, 2023

(7)

where c_j, $c_j'(j = 1, \ldots\ldots, m)$ are arbitrary constants.

3) Let now all the roots $\lambda_i \in \mathbb{Z}_-$ be simple. Then, the solution of equation (2) has the following form:

$$y(x) = \sum_{i=1}^{l} c_i\, p\left(\frac{1}{x-\lambda_i}\right), \tag{8}$$

where $c_i, (i = 1, \ldots\ldots, l)$ are arbitrary constants.

4) Finally, let the roots λ_i of the polynomial $P_l(\lambda)$ be the complex conjugates of an $\mu-$order of multiplicity such that $\lambda_i = \alpha \pm i\beta$.

Then, the solution of the considered equation corresponding to this root is defined by the following formula:

$$y(x) = \sum_{i=1}^{\mu} c_i\, x^{\alpha}(lnx)^{i-1}cos(\beta lnx)\theta(x) + \sum_{i=1}^{\mu} c_i'|x|^{\alpha}\,(ln|x|)^{i-1}sin(\beta ln|x|)\theta(-x), \tag{9}$$

where c_i, c_i' $(i = 1, \ldots\ldots, \mu)$ $-$ are arbitrary constants.

Immediately, we note that the verification of all these solutions under substitution into equation (2) has been realized in the simple case investigated in our published paper related to the second-order similar Euler differential equation. Refer to [17], [18].

All that has been investigated in the previous lines and on the basis of our results published in the paper, [17], leads us finally to write the general classical solution of equation (2) in the Euler case situation. So, takes place the following two theorems:

Theorem3.1.

Let $\prod_{i=0}^{l} a_i \neq 0,\ k_i \in \mathbb{N},\ i = 1, \ldots, l, k_0, s \in \mathbb{N} \cup \{0\}$ and fulfilled the condition

$s\ k_i = k_0 + i,\ i = 0, \ldots\ldots, l,\ \ \sum_{i=0}^{l}(-1)^i a_i\,(s + k_0 + i)! \neq 0$

Under the accomplishment of these conditions, the general solution of equation (1) can be expressed within the following part taking into consideration the nature of the roots of the polynomial $P_l(\lambda)$. So that.

1) Let $\forall\ j\ \in \mathbb{Z}_+, \sum_{i=0}^{l}(-1)^i a_i\,(j + k_0 + i)! \neq 0.$

a) The roots λ_i of the polynomial $P_l(\lambda)$ are simple and such that, all $\lambda_i > -1$ $(i = 1, l)$. Then the general solution of equation (1) has the following form:

$$y(x) = \frac{(-1)^{k_0}s!}{\sum_{i=0}^{l}(-1)^i a_i (s+k_0+i)!}\,\delta^{(s+k_0)}(x) + \sum_{j=0}^{k_0-1} C_j\ \delta(x)^{(j)} + \sum_{i=1}^{l} c_i\, x^{\lambda_i}\theta(x) + \sum_{i=1}^{l} c_i'\,|x|^{\lambda_i}\ \theta(-x), \tag{10}$$

where $c_0, \ldots\ldots, c_{k_0-1}; c_i, c_i'$ $(i = 1, l) -$ are arbitrary constants.

b) The roots of the polynomial $P_l(\lambda)$ are such that, λ_i are m_i $-$ order of multiplicity $(i = 1,2, \ldots\ldots, q \leq l)$, , and $\lambda_i > -1.$ Then, the solution is defined by:

$$y(x) = \frac{(-1)^{k_0}s!}{\sum_{i=0}^{l}(-1)^i a_i (s + k_0 + i)!}\,\delta^{(s+k_0)}(x) + \sum_{j=0}^{k_0-1} C_j\,\delta(x)^{(j)} + \sum_{i=1}^{q}\sum_{j=1}^{m_i} C_{ij}\, x^{\lambda_i}(lnx)^{j-1}\theta(x) + \sum_{i=1}^{q}\sum_{j=1}^{m_i} C_{ij}'\ |x|^{\lambda_i}(ln|x|)^{j-1}\theta(-x), \tag{11}$$

where $c_0, \ldots\ldots, c_{k_0-1}; c_{ij}, c_{ij}'$ $(j = 1,\ m_i) -$ are arbitrary constants.

c) The roots of the polynomial $P_l(\lambda)$ are such that, $\lambda_i \in \mathbb{Z}_-$ and simple. Then the solution is defined by the next formula :

$$y(x) = \frac{(-1)^{k_0}s!}{\sum_{i=0}^{l}(-1)^i a_i (s+k_0+i)!}\,\delta^{(s+k_0)}(x) + \sum_{j=0}^{k_0-1} C_j\ \delta(x)^{(j)} + \sum_{i=1}^{l} c_i\, p\left(\frac{1}{x-\lambda_i}\right), \tag{12}$$

where $c_0, \ldots\ldots, c_{k_0-1};\ c_i$ $(i = 1,2, \ldots\ldots, l) -$ are arbitrary constants.

d) The roots λ_i of the polynomial $P_l(\lambda)$ are complex conjugates of μ_i $-$order of multiplicity. Then, the solution is expressed by the next analytical formula:

$$y(x) = \frac{(-1)^{k_0}s!}{\sum_{i=0}^{l}(-1)^i a_i (s+k_0+i)!}\,\delta^{(s+k_0)}(x) + \sum_{j=0}^{k_0-1} C_j\,\delta(x)^{(j)} + \sum_{i=1}^{q}\sum_{j=1}^{\mu_i} c_{ij}\, x^{\alpha}(lnx)^{j-1}cos(\beta lnx)\theta(x) + \sum_{i=1}^{q}\sum_{i=1}^{\mu_i} c_{ij}'\,|x|^{\alpha}\,(ln|x|)^{j-1}sin(\beta ln|x|)\theta(-x), \tag{13}$$

where

$c_0, \ldots\ldots, c_{k_0-1}; c_{ij}, c_{ij}'$ $(j = 1, \ldots\ldots,\ \mu_i) -$are arbitrary constants.

Now let us move to the next following theorem.

Theorem3.2.

Let $\prod_{i=0}^{l} a_i \neq 0, k_i \in \mathbb{N}, i = 1, \ldots, l, k_0, s \in \mathbb{N} \cup \{0\}$ and fulfilled the conditions $k_i = k_0 + i, i = 0, \ldots\ldots, l,$ $\sum_{i=0}^{l}(-1)^i a_i\,(s + k_0 + i)! \neq 0,$ and be existing at least one

$j_*^m \in \mathbb{Z}_+/\{s\}, m \leq l,$ such that $j_*^m \in Nul\ P_l(j).$

Then, the general solution of the equation (1) is defined in the space K' by the formula:

a) The roots λ_i of the polynomial $P_l(\lambda)$ are simple and such that, $\lambda_i > -1$ $(i = 1,2, \ldots, l)$:

$$y(x) = \frac{(-1)^{k_0}s!}{\sum_{i=0}^{l}(-1)^i a_i (s+k_0+i)!}\,\delta^{(s+k_0)}(x) +$$

$\sum_{j=0}^{k_0-1} C_j\,\delta(x)^{(j)} + \sum_{j_*^m \in Nul\,P_l(j)} C_{j_*^m+k_0}\,\delta(x)^{(j_*^m+k_0)} + \sum_{i=1}^{l} c_i\,x^{\lambda_i}\,\theta(x) + \sum_{i=1}^{l} c_i{}'\,|x|^{\lambda_i}\,\theta(-x),$ (14)

where $c_0,\ldots\ldots,c_{k_0-1}, C_{j_*^m+k_0}(m \le l)$, $c_i, c_i{}'\ (i=1,l) -$ are arbitrary constants.

b) The roots λ_i of the polynomial $P_l(\lambda)$ are such that λ_i are $m_i -$ order of multiplicity $i=1,2,\ldots\ldots,q \le l$, and $\lambda_i > -1$. Then, the solution is defined by:

$$y(x) = \frac{(-1)^{k_0}s!}{\sum_{i=0}^{l}(-1)^i a_i (s+k_0+i)!}\,\delta^{(s+k_0)}(x) + \sum_{j=0}^{k_0-1} C_j\,\delta(x)^{(j)} + \sum_{j_*^m \in Nul\,P_l(j)} C_{j_*^m+k_0}\,\delta(x)^{(j_*^m+k_0)} + \sum_{i=1}^{q}\sum_{j=1}^{m_i} c_{ij}\,x^{\lambda_i}(ln x)^{j-1}\theta(x) + \sum_{i=1}^{q}\sum_{i=1}^{m_i} c_{ij}{}'|x|^{\lambda_i}\,(ln|x|)^{j-1}\theta(-x), \quad (15)$$

where $c_0,\ldots\ldots,c_{k_0-1}$, $C_{j_*^m+k_0}(m \le l)$, $c_{ij}, c_{ij}{}'\ (i=1,m_i)$ are arbitrary constants.

c) The roots λ_i of the polynomial $P_l(\lambda)$ are simple and such that $\lambda_i \in \mathbb{Z}_-$ and in this case the general solution of equation (1) is defined as follows:

$$y(x) = \frac{(-1)^{k_0}s!}{\sum_{i=0}^{l}(-1)^i a_i (s+k_0+i)!}\,\delta^{(s+k_0)}(x) + \sum_{j=0}^{k_0-1} C_j\,\delta(x)^{(j)} + \sum_{j_*^m \in Nul\,P_l(j)} C_{j_*^m+k_0}\,\delta(x)^{(j_*^m+k_0)} + \sum_{i=0}^{l} c_i\,p\left(\frac{1}{x^{-\lambda_i}}\right), \quad (16)$$

where $c_0,\ldots\ldots,c_{k_0-1}$, $C_{j_*^m+k_0}(m \le l)$, $c_i\ (i=1,l) -$ are arbitrary constants.

d) The roots λ_i of the polynomial $P_l(\lambda)$ are complex conjugates of $\mu_i -$ order of multiplicity $(\mu_i, i=1,2,\ldots,q,\le l)$. Then, the solution is expressed by the next analytical formula:

$$y(x) = \frac{(-1)^{k_0}s!}{\sum_{i=0}^{l}(-1)^i a_i (s+k_0+i)!}\,\delta^{(s+k_0)}(x) + \sum_{j=0}^{k_0-1} C_j\,\delta(x)^{(j)} + \sum_{j_*^m \in Nul\,P_l(j)} C_{j_*^m+k_0}\,\delta(x)^{(j_*^m+k_0)} + \sum_{i=1}^{q}\sum_{j=1}^{\mu_i} c_{ij}\,x^{\alpha}(ln x)^{j-1}cos(\beta ln x)\theta(x) + \sum_{i=1}^{q}\sum_{j=1}^{\mu_i} c_{ij}{}'\,|x|^{\alpha}\,(ln|x|)^{j-1}sin(\beta ln|x|)\theta(-x), \quad (17)$$

where $c_0,\ldots\ldots,c_{k_0-1}, C_{j_*^m+k_0}(m \le l), c_{ij}, c_{ij}{}'\ (j=1,\ldots\ldots,\mu_i) -$ are arbitrary constants.

<u>**Proof**</u>: The proofs of all these theorems can be easily deduced from the construction process even from the solutions obtained, analogically as done in the simple case investigated with full detail, [17].

Lastly, let us do this important remark also.

Remark. As we have completely investigated the differential equation (1) in the Euler case situation, then analogically as the results obtained from the investigation of the second order singular differential equation in the left Euler case situation, we can similarly consider the question of the solvency of the equation of the higher order for generalizing the general similar cases when considering the next differential equation: $\sum_{i=0}^{l} a_i\,x^{k_i}y^{(i)}(x) = \delta^{(s)}(x)$, with the conditions $k_{i+1} = k_i + 1;\ (i=0,1,\ldots\ldots,l-2)$ but $k_l \ne k_{l-1} + 1$. We may call this equation (in the two situations appearing) the *left Euler case equation* or the *right Euler case equation*, depending on the cases when $k_l > k_{l-1} + 1$ or if $k_l < k_{l-1} + 1$.

Before concluding, let's say that the generalized function $y(x) \in K'$ is a solution of the homogeneous differential equations (1)-(2) in the sense of distributions if and only if:

$$\forall\varphi(x) \in K, \left(\sum_{i=0}^{l} a_i\,x^{k_i}y^{(i)}(x), \varphi(x)\right) = (0, \varphi(x)) = 0. \quad (18)$$

And in the case of the non-homogeneous differential equation, $\sum_{i=0}^{l} a_i\,x^{k_i}y^{(i)}(x) = \delta(x)^{(s)}$ it should be realized: $y(x) \in K'$,

$$\forall\varphi(x) \in K, \left(\sum_{i=0}^{l} a_i\,x^{k_i}y^{(i)}(x), \varphi(x)\right) = \left(\delta(x)^{(s)}, \varphi(x)\right) = (-1)^s\varphi_{(0)}^{(s)} \quad (19)$$

For example, one should verify in the case when the roots of λ_i the polynomial $P_l(\lambda)$ are complex conjugate of $\mu_i\ (i=1,2,\ldots,q,\le l) -$order multiplicity, it should be realized

$$\left(\sum_{i=0}^{l} a_i\,x^{k_i}y^{(i)}(x), \varphi(x)\right) = \left(\delta(x)^{(s)}, \varphi(x)\right) = (-1)^s\varphi_{(0)}^{(s)}$$

with the solution expressed by the next formula obtained in (17) which we recall once:

$$y(x) = \frac{(-1)^{k_0} s!}{\sum_{i=0}^{l}(-1)^i a_i (s+k_0+i)!}\ \delta^{(s+k_0)}(x) + \sum_{j=0}^{k_0-1} C_j\ \delta(x)^{(j)} + \sum_{j_*^m \in Nul\ P_l(j)} C_{j_*^m+k_0}\ \delta(x)^{(j_*^m+k_0)} + \sum_{i=1}^{q}\sum_{j=1}^{\mu_i} c_{ij}\, x^{\alpha}(lnx)^{j-1}cos(\beta lnx)\theta(x) + \sum_{i=1}^{q}\sum_{j=1}^{\mu_i} c_{ij}{}'\ |x|^{\alpha}\ (ln|x|)^{j-1}sin(\beta ln|x|)\theta(-x).$$

Now let us state a conclusion at the end of this work.

IV. Conclusion

In this completed research, we have completely undertaken and finalized the search for classical solutions to the singular linear differential equation of Euler type (1) in the space of K' distributions. The methodology of the search for solutions is based on the replacement of the particular known form of the latter in the form $y(x) = x^{\lambda}$ (where λ —unknown value) in the equation in question, this led us to analyze the different roots of the characteristic polynomial $P_l(\lambda)$, and, depending on their nature, we managed to immediately write the form of the corresponding solutions in the sense of the distributions. By making an extension of this work and based on our research carried out in the past, of which we use the results obtained, we had formulated thus, in the global theorems 3.1 and 3.2 which describe, case by case, the general solutions in the sense of the distributions of the non-homogeneous equation $\sum_{i=0}^{l} a_i\, x^{k_i} y^{(i)}(x) = \delta(x)^{(s)}$ having the same form of the associated homogeneous equation of type (1).

It should be also remembered that in classical mathematical analysis, the notion of distribution, also called generalized function, refers to an object that generalizes the notion of function and measure. A well-developed theory of distributions establishes and extends the notion of derivative to all locally integrable functions and beyond. Note its use in the formulation of solutions to certain types of partial differential equations. Their notorious importance in the fields of physical sciences and engineering where there are several discontinuous problems naturally lead to situations of differential equations whose solutions are sought are distributions rather than ordinary functions.

V. Recommendations

This achieved work will help us to undertake in the brief future, the investigation of all distributional solutions of a more general standard situation of the *l-order* linear singular non-homogeneous differential equation in the cases called the left and the right *Euler* cases (in the space of generalized function K' of the following type $\sum_{i=0}^{l} a_i\, x^{k_i} y^{(i)}(x) = \sum_{s=0}^{n} \delta(x)^{(s)}$, when the parameters k_i satisfy the conditions $k_i = k_0 + i;\ i = 0,1\ldots\ldots, l$ and, for which we have already defined all the generalized - function solutions in the space K'. On this matter refer to the publications, [16], [17]. Therefore, and consequently, we could also completely be able to describe both all the generalized functions and classical regular solutions of the equation evoked.

References

[1] Kananthai, A. Distribution solutions of third order Euler equation. Southeast Asian Bull. Math.1999, 23,627-631.

[2] Gelfand. I.M., SHYLOV G.E: Space of Test and generalized functions. V.1.M.: Phyzmatgiz,1958. 307 Pages. Russian Edition.

[3] *(text in French)* L. Schwartz, *Théorie des distributions*, Tomes I et II, Hermann, Paris, 1957.

[4] Liangprom, A.; Nonlaopon, K. On the generalized solutions of certain fourth order Euler equations. J.Nonlinear Sci. Appl. 2017, 10, 4077- 4084.

[5] *(text in German)* Fuchs, *Zur theorie der linearen differentialgleichungen mit veränderlichen coefficienten,* Journal für die reine und angewandte mathematik, t.66, 1866, p.121-160.

[6] Abdourahman, *On a linear differential equation in the spaces of generalized functions*. Rostov on Don. Preprint at VINITI 02.08.2000. No.2035, 2000. 27 pages.

[7] Wiener, J.; Shah, S.M. Distributional and entire solutions of ordinary differential and functional differential equations. Int. J. Math. Sci. 1983, 6, 243–270.

[8] Kananthai, A. The distributional solutions of ordinary differential equation with polynomial coefficients. Southeast Asian Bull. Math. 2001,25,129-134.

[9] M. Gelfand, G. E. Shilov. “Generalized Functions,” Vol. II, Spaces of Fundamental and Generalized Functions. Academic Press. New York. 1968.

[10] Kanwal, R.P. Generalized functions: Theory and Technique, 3rd ed.; Springer: New York, NY, USA, 2004.

[11] Abdourahman. On a linear differential equation with singular coefficients. Rostov-On-Don. 1999. Collection of papers. «Integro-differential operators and their Applications» Rostov-Na-Donu: Izdat.DGTU,1999.N°.5.PP.4-7.Available at http://chat.ru/volume1999.

[12] Amornrat Sangsuwan∗, Kamsing Nonlaopon and Somsak Orankitjaroen. The generalized solutions of a certain *n-th* order Cauchy–Euler equation. Asian-European Journal of Mathematics Vol. 13, No. 1 (2020) 2050047 (11 pages) World Scientific Publishing Company. DOI: 10.1142/S1793557120500473.

[13] S. Jhanthanam, K Nonlaopon, and S Orankitjaroen, Generalized Solutions of the Third-Order Cauchy-Euler Equation in the Space of Right-Sided Distributions via Laplace Transform, Mathematics 2019, 7, 376; doi:10.3390/math7040376.

[14] Abdourahman. On a linear singular differential equation with singular coefficients, Collection of papers.” Integro-differential operators and their applications”, Rostov-na-Donu: Izdat. DGTU, 2001. No.5. P.4-10.

[15] JOSEPH WIENER. Generalized-Function Solutions of Differential and Functional Differential Equations. Journal of Mathematical Analysis and Applications 88, 170-182 (1982). Edinburg, Texas 78539.

[16] Abdourahman. Generalized-function Solutions of a Differential Equation of L-order in the Space K'. *International Journal of Theoretical and Applied Mathematics*. Vol. 8, No. 2, 2022, pp. 40-44. doi: 10.11648/j.ijtam.20220802.12.

[17] Abdourahman. On Distributional Solutions of a Singular Differential Equation of 2-order in the Space K'. *International Journal of Theoretical and Applied Mathematics*. Vol. 8, No. 2, 2022, pp. 45-50. doi: 10.11648/j.ijtam.20220802.13.

[18] Abdourahman. On generalized-function solutions of a first-order linear singular differential equation in the space K' via Fourier transform. Journal of Mathematical Sciences: Advances and Applications.Volume 70, 2022, Pages 27-51. Available at http://scientificadvances.co.in.DOI:ttp://dx.doi.org/10.18642/jmsaa_7100122244.

[19] Amphon Liangprom, Kamsing Nonlaopon. On the generalized solutions of certain fourth-order Euler equations. Available online at www.isr-publications.com/jnsa J.Nonlinear Sci.Appl.,10(2017),4077-4084. Research Article ISSN:2008-1898.

[20] Abdourahman. On a linear singular differential equation with singular coefficients in the space of generalized functions. Part II, UDK UDK 517.938; 517.95. Rostov-On-Don, 2001. Preprint at VINITI Moscou.

[21] R. P. Kanwal, Generalized Functions: Theory and Technique, 3rd ed. (Springer Science & Business Media, 2004).

[22] Kananthai and K. Nonlaopon, On the weak solution of the compound ultra-hyperbolic equation, Chang Mai J. Sci. 1 (2002) 209–214.

[23] Kananthai, S. Suantai, and V. Longani, On the weak solutions of the equation related to the diamond operator, Vychisl. Tekhnol. 5 (2000) 61–67.

[24] Kananthai, The distributional solutions of ordinary differential equation with polynomial coefficients, Southeast Asian Bull. Math. 25 (2001) 129–134.

[25] Kananthai, Distribution solutions of the third order Euler equation, Southeast Asian Bull. Math. 23 (1999) 627–631.

[26] H. Kim, the solution of Euler–Cauchy equation expressed by didifferential perator usi Laplace transform, Int. J. Pure Appl. Math. 84 (2013) 345–351.

[27] A.M. Krall, R. P. Kanwal and L. L. Littlejohn, Distributional solutions of ordinary differential equations, Canadian Math. Soc. Conf. Proc. (1987) 227–246.

Contribution of individual authors to the creation of a scientific article (ghostwriting policy)

Abdourahman HAMAN ADJI had set the problem investigated and proposed the methodology of the research conducted within the work with all computations.

SHANKISHVILI Lamara Dmitrievna verified some computations realized within the work and proposed the main idea of their presentation.

Sources of funding for research presented in a scientific article or scientific article itself

The realization of this research work partially started at the Chair of Differential and Integral Equations of the Faculty of Mechanics and Mathematics of the State University of Rostov in the Russian Federation and finalized at the University of Maroua in Cameroon is financed by the premium for the modernization of research from the Ministry of Higher Education of Cameroon.

Conflict of Interest Statement:

This is to affirm clearly that there is no conflict of interest amongst the authors or whosoever.