# General Periodic Functions and Generalization of Fourier analysis

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**Abstract — In this paper, we consider $l_p$-periodical functions $pcs(m\theta)$ and $psn(m\theta)$, which are defined on a curve given by an equation $|x|^p+|y|^p=1$ on $R^2$ as functions of its length. Considering $pcs(m\theta)$ and $psn(m\theta)$ as an independent functional system, we construct the theory similar to Fourier analysis with the proper weights. For these weights, we establish an analog of the Riemannian theorem. The adjoint representations are introduced and dual theory is developed. These Fourier representations can be used for approximation for the oscillation processes.**

**Keywords— General periodic function, Fourier analysis, p-circle, adjoint, p-Laplacian, linear approximation, spectral theory, oscillation.**

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## I. Introduction

The main motivation of this paper is to generalize the methods of the Fourier analysis for the measurable functions and methodology of the Fourier series, which presents a possibility to express a function as a sum of functions depending on frequency. Applying the Fourier series, we can represent a function as the sum of sines and cosines defined of the unit circle.

The theoretical foundations of Fourier representation were developed in the fundamental works of classical functional analysis. For the most recent advances in the related theoretical studies and their applications, the interested reader is referred to the references below.

The curve defined by the equation $|x|^p+|y|^p=1$ will be called a unit p-circle on a plane $R^2$. In this paper, we construct a pair of $l_p$-periodical functions defined on the p-circle, which, in case $p=2$, coincide with the elemental trigonometric functions. We introduced an approximation of arbitrary measurable function $f$ from $L^p$ by the summation of the linear combination of the special $l_p$-periodical functions $pcs(m\theta)$ and $psn(m\theta)$ with the proper weights, which are being obtained as integrals of $f$. For these weights, we establish an analog of the Riemannian theorem. Also, the adjoint theory has been developed. We establish the representation of the functions $f\in L^p\left[0,l_p\right]$ in the form of the series as

$$f(x)=\frac{1}{l_p}\int_0^{l_p} f(x)\,dx+$$

$$\frac{2}{l_p}\sum_{m=1,2,\dots}\int_0^{l_p} f(y)\,pcs(my)\left|pcs(my)\right|^{p-2}pcs(mx)\,dy+$$

$$\frac{2}{l_p}\sum_{m=1,2,\dots}\int_0^{l_p} f(y)\,psn(my)\left|psn(my)\right|^{p-2}psn(mx)\,dy$$

and its adjoint

$$f(x)\left|f(x)\right|^{p-2}=\tilde a_0+\sum_{m=1,2,\dots}\begin{pmatrix}\tilde a_m\,pcs(mx)\left|pcs(mx)\right|^{p-2}+\\ \tilde b_m\,psn(mx)\left|psn(mx)\right|^{p-2}\end{pmatrix},$$

where $\tilde a_0,\tilde a_m,\tilde b_m$ are adjoint weights.

Let us assume $\varphi$ and $\psi$ is a pair of smooth functions of the real argument and such that the following two conditions

$$\varphi(x)\psi(x)=\varphi(\psi(x))$$
$$f(g(x))=g(f(x))$$

are satisfied for all $x \in R^1$. Then, the pair $\varphi$ and $\psi$ can be chosen as $\varphi(\cdot) = |\cdot|^p$ and $\psi(\cdot) = |\cdot|^q$ under the conditions $\frac{p}{p-1} = q, \quad 1<p$. These properties of the exponential functions render us the following properties of the class $A_p$.

We can define the functional class $A_p$ as the class of all weights such that

$$\left( \frac{1}{mes(B)} \langle |f| \rangle_B \right)^p \le \frac{const}{\langle \omega \rangle_B} \left\langle \omega |f|^p \right\rangle_B$$

holds for an arbitrary locally integrable function $f$ and any ball $B$.

For any measure $d\mu(x) = \omega(x)dx$ for any $\omega \in A_p$, the maximal operator $M$ on $R^n$ satisfies the following estimation

$$\left\langle |Mf|^p \right\rangle_{d\mu} \le A \left\langle |f|^p \right\rangle_{d\mu},$$

and if $T$ is calderon zygmund operator under standard conditions, we have

$$\left\langle \omega(x) \left( \sup_{\varepsilon>0} \left| \left\langle K(x,y) f(y) \right\rangle_{dy,\ |x-y|>\varepsilon} \right| \right)^p \right\rangle_{dx} \le$$

$$\le C_{p,\omega} \left\langle |Mf(x)|^p \omega(x) \right\rangle_{dx}$$

for any $f \in L^2$.

Exploiting the functions $pcs(\theta)$ and $psn(\theta)$, we define the p-sphere, this p-sphere has p-spherical symmetry but it cannot rotate on itself and its smoothness depends on $p$ as a parameter. The topology of this sphere is collapsing when $p$ approaches one or infinity. If $p=1$ there are four critical points, vertexes in which even first smoothness is absent, a situation is similar when $p \to \infty$.

The presented constructive theory provides the methodology for the researchers to develop a plethora of computational methods, which can be applied in different fields for obtaining approximations of the solution to many mathematical models of physical processes.

## II. Definitions and Notations

For $p \in (1, \infty)$, on the real two-dimensional plane, let us consider a curve given by equation

$$|x|^p + |y|^p = 1. \tag{1}$$

If $p=2$ this curve is a two-dimensional circle, let us denote its length as $l_2$ equals $2\pi_2$. If $p \ne 2$ this curve is not invariant under rotation and we denote its length as $l_p$ that can be written in the form of the following integral

$$\int_0^{2\pi_2} \frac{2}{p} \left( \sqrt{ \begin{matrix} (\cos(\theta))^{\frac{4}{p}-2} (\sin(\theta))^2 + \\ + (\sin(\theta))^{\frac{4}{p}-2} (\cos(\theta))^2 \end{matrix} } \right) d\theta = 2\pi_p, \tag{2}$$

where $\pi_p$ is a real constant obtained from this integral as a parameter of $p>1$.

Let us assume that we want to measure the distance as $r = \sqrt[p]{|x|^p + |y|^p}$ then the metric of such system is $g_{ik} = \frac{4}{p^2} \begin{pmatrix} \alpha^{\frac{4}{p}-2} & 0 \\ 0 & \beta^{\frac{4}{p}-2} \end{pmatrix}$ and (1) becomes equation of the unit circle in new metric.

We consider a pair of $C^1$-smooth functions $pcs(\theta)$ and $psn(\theta)$ of the real argument $\theta$ and such that:

$$|psn(\theta)|^p + |pcs(\theta)|^p = 1 \quad \text{for all} \quad \theta \in R^1. \tag{3}$$

We define these functions as values of coordinates $x$ and $y$, respectively, dependent on the length of the curve $l_p$ as a parameter, and taking $pcs(0)=1$ by definition, so that

$$pcs(\theta) = x \quad \text{for all} \quad \theta \in R^1 \tag{4}$$

$$psn(\theta) = y \quad \text{for all} \quad \theta \in R^1. \tag{5}$$

If $p=2$ we have $psn(\theta) = \sin(\theta)$ and $pcs(\theta) = \cos(\theta)$ for all $\theta \in R^1$.

From the definition, we establish that functions $psn(\theta)$ and $pcs(\theta)$ are $l_p$-periodical and such that $psn(0) = pcs\left(\frac{l_p}{4}\right) = 0$, $pcs(0) = psn\left(\frac{l_p}{4}\right) = 1$ and $pcs\left(\frac{l_p}{8}\right) = psn\left(\frac{l_p}{8}\right) = \frac{1}{\sqrt[p]{2}}$, and next $l_p$-periodically. The smoothness of $psn(\theta)$ and $pcs(\theta)$ functions depends on $p$.

Let us denote $\rho = \sqrt[p]{|x|^p + |y|^p}$ and $\varphi = \theta, \quad \varphi \in [0, l_p]$ such that

$$x = \rho\ pcs(\theta)$$
$$y = \rho\ psn(\theta),$$

the parameters $\rho$ and $\varphi$ can be taken as a new coordinate system similar to the polar coordinate system. Together with the system $(\rho, \varphi)$, we can consider the dual system as

$$x = \rho|\rho|^{p-2}\ pcs(\theta)|pcs(\theta)|^{p-2}$$
$$y = \rho|\rho|^{p-2}\ psn(\theta)|psn(\theta)|^{p-2}.$$

Let us assume that $q = \dfrac{p}{p-1}$ then the last system defines a "circle" in the metric $r = \sqrt[q]{|x|^q + |y|^q}$, with $r = \rho^p$ is a "circle" for any fixed value of $\rho$.

Similar, for any given $\rho$ we can consider a sphere n-dimensional sphere

$$|x^1|^p + |x^2|^p + |x^3|^p + \dots + |x^{n-2}|^p +$$
$$|x^{n-1}|^p + |x^n|^p = \rho^p, \quad \rho > 0$$

in the form

$$x^1 = \rho\ pcs(\theta^1)$$
$$x^2 = \rho\ psn(\theta^1)\, pcs(\theta^2)$$
$$x^3 = \rho\ psn(\theta^1)\, psn(\theta^2)\, pcs(\theta^3)$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$x^{n-1} = \rho\ psn(\theta^1)\, psn(\theta^2) \dots psn(\theta^{n-2})\, pcs(\theta^{n-1})$$
$$x^n = \rho\ psn(\theta^1)\, psn(\theta^2) \dots psn(\theta^{n-2})\, psn(\theta^{n-1}),$$

where $\theta^1, \dots, \theta^{n-2} \in \left[0, \dfrac{l}{2}_p\right], \quad \theta^{n-1} \in [0, l_p]$.

The adjoint sphere is defined by the following formula

$$|x^1|^q + |x^2|^q + |x^3|^q + \dots + |x^{n-2}|^q +$$
$$|x^{n-1}|^q + |x^n|^q = \rho^q, \quad \rho > 0$$

and in a parametric form

$$x^1 = \rho\ pcs(\theta^1)|pcs(\theta^1)|^{p-2}$$
$$x^2 = \rho\ psn(\theta^1)|psn(\theta^1)|^{p-2}\ pcs(\theta^2)|pcs(\theta^2)|^{p-2}$$
$$x^3 = \rho\ psn(\theta^1)|psn(\theta^1)|^{p-2}\ pcs(\theta^3)|pcs(\theta^3)|^{p-2}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$x^{n-1} =$$
$$\rho\ psn(\theta^1)|psn(\theta^1)|^{p-2} \dots pcs(\theta^{n-1})|pcs(\theta^{n-1})|^{p-2}$$
$$x^n =$$
$$\rho\ psn(\theta^1)|psn(\theta^1)|^{p-2} \dots psn(\theta^{n-1})|psn(\theta^{n-1})|^{p-2},$$

where $q = \dfrac{p}{p-1}$.

The equation of p-torus in $R^3$ space takes the form

$$x = (A + B\ pcs(\theta^1))\rho\ pcs(\theta^2)$$
$$y = (A + B\ pcs(\theta^1))\rho\ psn(\theta^2)$$
$$z = B\ psn(\theta^2), \quad \theta^1, \dots, \theta^2 \in [0, l_p];$$

indeed, this equation can be presented in the form

$$\left(\left(|x|^p + |y|^p\right)^{\frac{1}{p}} - A\right)^p + |z|^p = B^p,$$

where $A, B$ are usual parameters.

Next, we establish

$$\frac{d}{d\theta} pcs(\theta) = -psn(\theta)|psn(\theta)|^{p-2} \text{ for all } \theta \in R^1 \quad (6)$$

and

$$\frac{d}{d\theta} psn(\theta) = pcs(\theta)|pcs(\theta)|^{p-2} \text{ for all } \theta \in R^1. \quad (7)$$

Assuming that the second derivatives exist, we can write

$$\frac{d^2}{d\theta^2} pcs(\theta) =$$
$$-(p-1)\, pcs(\theta)|psn(\theta)|^{p-2}|pcs(\theta)|^{p-2} \quad \forall \theta \in R^1$$
(8)

$$\frac{d^2}{d\theta^2} psn(\theta) =$$
$$-(p-1)\, psn(\theta)|psn(\theta)|^{p-2}|pcs(\theta)|^{p-2} \quad \forall \theta \in R^1.$$
(9)

Also, there is an obvious integral identity

$$psn(\theta)\, pcs(\theta) = \int \left((pcs(\theta))^p - (psn(\theta))^p\right) d\theta.$$
(10)

## III. Approximation

Mappings $pcs(\theta)$ and $psn(\theta)$ are $l_p$-periodic functions of the real argument $\theta$, however, functions

$psn(\theta)$ and $pcs(\theta)$ are dependent also on $p$ as a parameter so that not only their values depend on $p$ but their period $l_p$ also. Each value $p$ corresponds to certain functions such if $p=2$ then $psn(\theta)=\sin(\theta)$, $pcs(\theta)=\cos(\theta)$ and $l_2=2\pi$.

Let us find a representation of the Lebesque real-valued measurable function $f(x)$ as a series with appropriate weights on the interval $[0, l_p]$

$$\begin{aligned} f(x) &= a_0 + (a_1 pcs(x) + b_1 psn(x)) + \\ &(a_2 pcs(2x) + b_2 psn(2x)) + \\ &+ (a_3 pcs(3x) + b_3 psn(3x)) + \ldots = \\ &= a_0 + \sum_{m=1,2,\ldots} (a_m pcs(mx) + b_m psn(mx)), \end{aligned} \quad (11)$$

where $a_0, a_1, b_1, \ldots, a_m, b_m, \ldots$ are some real coefficients, which must be found.

Integrating the identity (3) over the period $l_p$ and taking into account (10), we obtain

$$\int_0^{l_p} |pcs(\theta)|^p \, d\theta = \int_0^{l_p} |psn(\theta)|^p \, d\theta = \frac{l_p}{2}.$$

Assume that function $f$ can be represented by (11) then we can (11) integrate over the interval $[0, l_p]$, period of $pcs(mx)$ *and* $psn(mx)$, so we can find the first coefficient $a_0$ as

$$a_0 = \frac{1}{l_p} \int_0^{l_p} f(x) dx. \quad (12)$$

To obtain a value of the coefficient $a_m$, we are multiplying the equality (11) by $pcs(nx)|pcs(nx)|^{p-2}$ and integrating over the period $[0, l_p]$, we have

$$a_m = \frac{2}{l_p} \int_0^{l_p} f(x)\, pcs(mx) |pcs(mx)|^{p-2} dx$$

(13)

since

$$\int_0^{l_p} |pcs(mx)|^p dx = \frac{2}{l_p}$$

for any $m > 0$, and

$$\int_0^{l_p} pcs(nx)\, pcs(mx) |pcs(mx)|^{p-2} dx = 0 \quad \text{for } n \neq m$$

,

and for all $m$ *and* $n$

$$\int_0^{l_p} psn(nx)\, pcs(mx) |pcs(mx)|^{p-2} dx = 0.$$

For finding coefficient $b_m$ we multiply (11) by $psn(nx)|psn(nx)|^{p-2}$ and integrating over $[0, l_p]$, we obtain

$$b_m = \frac{2}{l_p} \int_0^{l_p} f(x)\, psn(mx) |psn(mx)|^{p-2} dx$$

(14)

since

$$\int_0^{l_p} |psn(mx)|^p dx = \frac{2}{l_p},$$

$$\int_0^{l_p} psn(nx)\, psn(mx) |psn(mx)|^{p-2} dx = 0 \quad \forall n \neq m$$

$$\int_0^{l_p} pcs(nx)\, psn(mx) |psn(mx)|^{p-2} dx = 0 \quad \forall m,\ n.$$

Formulae (12) – (14) are similar to the formulae of the Fourier coefficients of the Hilbert space theory.

Thus, we obtain the mapping of the functions $f \in L^p[0, l_p]$ in the set of the infinite series according to the formula

$$f(x) \sim \frac{1}{l_p} \int_0^{l_p} f(x) dx +$$

$$\frac{2}{l_p} \sum_{m=1,2,\ldots} \left( \begin{aligned} &\int_0^{l_p} f(y)\, pcs(my) |pcs(my)|^{p-2} pcs(mx) + \\ &+ \int_0^{l_p} f(y)\, psn(my) |psn(my)|^{p-2} psn(mx) \end{aligned} \right) dy.$$

(15)

**Theorem (analog Riemannian theorem) 1.** Assuming $g$ is an integrable function over an arbitrary interval $[a, b] \subset R^1$ then there are

$$\lim_{m\to\infty} \int_a^b g(x)\, psn(mx) |psn(mx)|^{p-2} dx = 0$$

(16)

and

$$\lim_{m\to\infty} \int_a^b g(x)\, pcs(mx) |pcs(mx)|^{p-2} dx = 0$$

(17)

**Proof**. Proving of (16) and (16) are similar to each other, so we are going to prove only (16). We have the following estimations

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$$\left|\int_a^b psn(mx)\left|psn(mx)\right|^{p-2}dx\right| = \left|\frac{1}{m}\int_a^b d\ \ pcsp(mx)\right| \le$$

$$\le \frac{1}{m}\left|pcsp(mb) - pcsp(ma)\right| \le \frac{2}{m}.$$

Let us split the interval $[a, b]$ into n-parts by points

$$a = x_0 < x_1 < ... < x_i < x_{i+1} < ... < x_n = b$$

so, we can split the integral into the sum of the integrals as

$$\int_a^b g(x)\, psn(mx)\left|psn(mx)\right|^{p-2}dx =$$

$$\sum_{i=1,...,n-1}\int_{x_i}^{x_{i+1}} g(x)\, psn(mx)\left|psn(mx)\right|^{p-2}dx.$$

This equality can be transformed as follows

$$\int_a^b g(x)\, psn(mx)\left|psn(mx)\right|^{p-2}dx =$$

$$= \sum_{i=1,...,n-1}\int_{x_i}^{x_{i+1}}\left(\begin{matrix} g(x) - \\ \inf\limits_{[x_i,\, x_{i+1}]} g(x)\end{matrix}\right) psn(mx)\left|psn(mx)\right|^{p-2}dx +$$

$$+ \sum_{i=1,...,n-1}\left(\inf_{[x_i,\, x_{i+1}]} g(x)\right)\int_{x_i}^{x_{i+1}} psn(mx)\left|psn(mx)\right|^{p-2}dx.$$

Next, we estimate

$$\int_a^b g(x)\, psn(mx)\left|psn(mx)\right|^{p-2}dx \le$$

$$\le \sum_{i=1,...,n-1}\left(g(x) - \inf_{[x_i,\, x_{i+1}]} g(x)\right)(x_{i+1} - x_i) +$$

$$+ \frac{2}{m}\sum_{i=1,...,n-1}\inf_{[x_i,\, x_{i+1}]} g(x).$$

Thus, for any $\varepsilon > 0$, we choose a partition in such a way that

$$\sum_{i=1,...,n-1}\left(g(x) - \inf_{[x_i,\, x_{i+1}]} g(x)\right)(x_{i+1} - x_i) < \frac{\varepsilon}{2},$$

(it always can be done, maybe except on a set $\Theta$ of a very small measure that limits to zero, and on this set, there is an estimation

$$\left|\int_\Theta g(x)\, psn(mx)\left|psn(mx)\right|^{p-2}dx\right| \le \int_\Theta \left|g(x)\right|dx,$$

then we consider that set $[a, b] - \Theta$ as an initial domain). Next, we take numbers $m$ large enough such that $m > \frac{4}{\varepsilon}\sum_{i=1,...,n-1}\inf_{[x_i,\, x_{i+1}]} g(x)$, for such $p$, we have

$$\left|\int_a^b g(x)\, psn(mx)\left|psn(mx)\right|^{p-2}dx\right| \le \varepsilon,$$

thus, the theorem is proven.

As a consequence of the theorem, we have that the coefficients $a_m$ and $b_m$ approach zero as $m$ approaches infinity.

We also can formulate an adjoint variant of theorem 1

**Theorem (adjoint) 2.** Let $g$ be an integrable function over an arbitrary interval $[a, b] \subset R^1$ then there are

$$\lim_{m\to\infty}\int_a^b g(x)\, psn(mx)\,dx = 0$$

and

$$\lim_{m\to\infty}\int_a^b g(x)\, pcs(mx)\,dx = 0.$$

## IV. THE ADJOINT SERIES

Let us consider a formal series

$$\tilde{a}_0 + \sum_{m=1,2,...}\left(\begin{matrix}\tilde{a}_m pcs(mx)\left|pcs(mx)\right|^{p-2} + \\ \tilde{b}_m psn(mx)\left|psn(mx)\right|^{p-2}\end{matrix}\right), \quad (18)$$

we will call this series an adjoint of the series (11).

Similarly to (11), we can represent the Lebesque measurable real-valued function $f\left|f\right|^{p-2}$ on the interval $[0, l_p]$ as

$$f(x)\left|f(x)\right|^{p-2} \sim \tilde{a}_0 + \sum_{m=1,2,...}\left(\begin{matrix}\tilde{a}_m pcs(mx)\left|pcs(mx)\right|^{p-2} + \\ \tilde{b}_m psn(mx)\left|psn(mx)\right|^{p-2}\end{matrix}\right), \quad (19)$$

where $\tilde{a}_0, \tilde{a}_1, \tilde{b}_1, ..., \tilde{a}_m, \tilde{b}_m, ...$ defined as follows

$$\tilde{a}_0 = \frac{1}{l_p}\int_0^{l_p} f(x)\left|f(x)\right|^{p-2}dx, \quad (20)$$

$$\tilde{a}_m = \frac{2}{l_p}\int_0^{l_p} f(x)\left|f(x)\right|^{p-2} pcs(mx)\,dx \quad (21)$$

and

$$\tilde{b}_m = \frac{2}{l_p}\int_0^{l_p} f(x)\left|f(x)\right|^{p-2} psn(mx)\,dx. \quad (22)$$

Let us denote partial sums of the series (11) and (18) as

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$$S_n(x)= a_{\mathbf{0}}+\sum_{m=\mathbf{1},\mathbf{2},\dots,n}\left(a_m pcs(mx)+b_m psn(mx)\right), \quad (23)$$

and

$$\tilde{S}_n(x)=\tilde{a}_{\mathbf{0}}+ \sum_{m=\mathbf{1},\mathbf{2},\dots,n}\left(\begin{matrix}\tilde{a}_m pcs(mx)\left|pcs(mx)\right|^{p-\mathbf{2}}+\\ \tilde{b}_m psn(mx)\left|psn(mx)\right|^{p-\mathbf{2}}\end{matrix}\right) \quad (24)$$

respectively.

We calculated the integrals

$$\int_{\mathbf{0}}^{l_p} f(x)\left|f(x)\right|^{p-\mathbf{2}} S_n(x)\,dx= a_{\mathbf{0}}\int_{\mathbf{0}}^{l_p} f(x)\left|f(x)\right|^{p-\mathbf{2}}\,dx+ +\sum_{m=\mathbf{1},\mathbf{2},\dots,n}\int_{\mathbf{0}}^{l_p} f(x)\left|f(x)\right|^{p-\mathbf{2}}\left(\left(\begin{matrix}a_m pcs(mx)+\\ b_m psn(mx)\end{matrix}\right)\right)dx= =a_{\mathbf{0}}\tilde{a}_{\mathbf{0}}l_p+\frac{l_p}{2}\sum_{m=\mathbf{1},\mathbf{2},\dots,n}\left(a_m\tilde{a}_m+b_m\tilde{b}_m\right) \quad (25)$$

and adjoint integrals

$$\int_{\mathbf{0}}^{l_p} f(x)\tilde{S}_n(x)\,dx=\tilde{a}_{\mathbf{0}}\int_{\mathbf{0}}^{l_p} f(x)\,dx+ +\sum_{m=\mathbf{1},\mathbf{2},\dots,n}\int_{\mathbf{0}}^{l_p} f(x)\left(\begin{matrix}\tilde{a}_m pcs(mx)\left|pcs(mx)\right|^{p-\mathbf{2}}+\\ \tilde{b}_m psn(mx)\left|psn(mx)\right|^{p-\mathbf{2}}\end{matrix}\right)dx= =a_{\mathbf{0}}\tilde{a}_{\mathbf{0}}l_p+\frac{l_p}{2}\sum_{m=\mathbf{1},\mathbf{2},\dots,n}\left(a_m\tilde{a}_m+b_m\tilde{b}_m\right), \quad (26)$$

and

$$\int_{\mathbf{0}}^{l_p} S_n(x)\tilde{S}_n(x)\,dx=a_{\mathbf{0}}\tilde{a}_{\mathbf{0}}\int_{\mathbf{0}}^{l_p}dx+ \sum_{m=\mathbf{1},\mathbf{2},\dots,n}\int_{\mathbf{0}}^{l_p}\left(\begin{matrix}a_m\tilde{a}_m\left|pcs(mx)\right|^{p}+\\ b_m\tilde{b}_m\left|psn(mx)\right|^{p}\end{matrix}\right)dx= =a_{\mathbf{0}}\tilde{a}_{\mathbf{0}}l_p+\frac{l_p}{2}\sum_{m=\mathbf{1},\mathbf{2},\dots,n}\left(a_m\tilde{a}_m+b_m\tilde{b}_m\right), \quad (27)$$

the coefficients $l_p$ and $\frac{l_p}{2}$ can be made equal to one by renormalizations.

Let us assume that function $f\in L^p$ then $f\left|f\right|^{p-\mathbf{2}}\in L^{\frac{p}{p-\mathbf{1}}}$ that the following integrals are correctly defined

$$\int_{\mathbf{0}}^{l_p}\left(\left(f(x)-S_n(x)\right)\left(f(x)\left|f(x)\right|^{p-\mathbf{2}}-\tilde{S}_n(x)\right)\right)dx= =\|f\|_p^p-\left(a_{\mathbf{0}}\tilde{a}_{\mathbf{0}}l_p+\frac{l_p}{2}\sum_{m=\mathbf{1},\mathbf{2},\dots,n}\left(a_m\tilde{a}_m+b_m\tilde{b}_m\right)\right). \quad (28)$$

Now, let us take function

$$f=a_{\mathbf{0}}+\sum_{m=\mathbf{1},\mathbf{2},\dots,n}\left(a_m pcs(mx)+b_m psn(mx)\right) \quad (29)$$

and multiplying (29) by $f\left|f\right|^{p-\mathbf{2}}$, we obtain

$$\|f\|_p^p=\left(a_{\mathbf{0}}\tilde{a}_{\mathbf{0}}l_p+\frac{l_p}{2}\sum_{m=\mathbf{1},\mathbf{2},\dots,n}\left(a_m\tilde{a}_m+b_m\tilde{b}_m\right)\right).$$

So, we have obtained the next theorem.

**Theorem 3.** ***The system of functions*** $\frac{1}{\sqrt{l_p}}$, $\frac{\sqrt{2}\ psn(m\,\theta)}{\sqrt{l_p}}$ ***and*** $\frac{\sqrt{2}\ pcs(m\,\theta)}{\sqrt{l_p}}$ ***is closed in*** $L^p\left(\left[0,\,l_p\right]\right)$.

## V. Application $pcs(\theta)$ and $psn(\theta)$ functions to differential systems

Since the functions $pcs(\theta)$ and $psn(\theta)$ have exponential derivatives, they can be applied to the solution of differential equations. On a real number line, let us consider a non-linear autonomic system of two equations

$$\frac{d}{d\theta}x(\theta)=-y(\theta)\left|y(\theta)\right|^{p-\mathbf{2}} \quad \text{for all} \quad \theta\in R^{\mathbf{1}}$$
$$\frac{d}{d\theta}y(\theta)=x(\theta)\left|x(\theta)\right|^{p-\mathbf{2}} \quad \text{for all} \quad \theta\in R^{\mathbf{1}},$$

here $2\le p$.

The general solution to this system can be written in the following form

$$x(\theta)=C_{\mathbf{2}}^{\frac{\mathbf{1}}{\sigma}}pcs(C_{\mathbf{1}}\theta),\quad \theta\in R^{\mathbf{1}}$$
$$y(\theta)=C_{\mathbf{1}}^{\frac{\mathbf{1}}{\sigma}}psn(C_{\mathbf{2}}\theta),\quad \theta\in R^{\mathbf{1}}.$$

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