A variational approach for estimating the temperature distribution in the body of a rectangular parallelepiped shape

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Abstract— The novelty of this study is a variational approach for estimating the temperature distribution in the body of a rectangular parallelepiped shape when a heat flow enters one of the faces of a rectangular parallelepiped, and heat exchange with the environment occurs on the opposite side. At the same time, options are considered when the remaining faces of a rectangular parallelepiped are thermally insulated or vice versa. My contribution of this work is that we calculated the laws of temperature distribution when dividing a rectangular parallelepiped into a different number of elements. It is shown that acceptable accuracy is achieved already by dividing the sides of a rectangular parallelepiped into three or 4 parts. In addition, a comparison of the temperature distribution law for a rectangular parallelepiped and a rod close in size, other things being equal, was carried out. Their slight difference is shown.

Keywords—Variational approach thermal conductivity, heat flow, rectangular parallelepiped, heat exchange.

I. Introduction

In [1], general problems on the use of the finite element method for determining the thermo-mechanical characteristics of various solids are considered.

In [2], a computational algorithm and a method for determining the temperature field along the length of a rod with a limited length and variable cross-section are proposed.

In [3], an energy method is considered for determining the law of temperature distribution, three components of deformation and stress, provided that both ends of a rod of variable cross-section are rigidly fixed.

In [4], the stationary solution of thermal conductivity problems with low convergence is shown for a rectangle with specified zero temperatures, with the exception of one surface with an abrupt temperature change.

In [5], exact nonstationary solutions of thermal conductivity in two-dimensional rectangles heated at the boundary are considered. The solution of the standard method of separation of variables consists of two parts: stationary and additional transient.

In [6], the basics of calculating heat transfer through a layer of matter are described. Solutions of the problems of determining heat flows through a layer of matter with different conditions at the boundaries of the layer and different properties of the substance of the layer are given.

[7] is devoted to numerical methods for solving problems of unsteady thermal conductivity taking into account the relaxation of the heat flow. A mathematical model based on a system of hyperbolic heat conduction equations is presented for calculating the temperature field in a two-layer infinitely extended plate with the conditions of conjugation of an ideal contact.

In [8], the equation of thermal conductivity of an eccentric spherical ring with an inner surface maintained at a constant temperature and an outer surface subject to convection is analytically solved.

In [9], one computational approach to the calculation of the heat equation is presented, which differs in the case of three-dimensional oblique unstructured grids by the compactness of the grid pattern and the unconditional stability of the numerical algorithm.

In [10], solutions to problems of nonstationary thermal conductivity (semi-bounded body, unbounded plate, solid cylinder, ball, hollow cylinder) are considered by several methods (separation of variables, operational, integral Fourier and Hankel transformations). The solutions are given in generalized variables using the method of similarity theory, they are illustrated by numerous graphs and tables.

Mathematical steps leading to the calculation of the temperature field in multidimensional multilayer bodies are described in [11] and numerical results for two-layer bodies are presented. Representations include boundary conditions of the first, second and third kind. An efficient computational scheme for calculating eigenvalues is discussed and numerical results are given.

In [12], the temperature distribution under the influence of a heat flow from one side of a parallelepiped and during heat exchange with the opposite side is considered using an approach where the solution of the problem is expressed as a solution of one-dimensional problems and multiple Fourier series or their generalization are used

In [13], a representative stationary problem of thermal conductivity in rectangular bodies with uniformly distributed heat generation is analytically investigated. A simple and accurate model is proposed that allows for the first time to predict the dimensionless parameter of the shape factor. The model is very concise and convenient for fast approximation to the real world, and also provides acceptable accuracy for engineering practice.

II. PROBLEM STATEMENT

Consider a solid body in the form of a rectangular parallelepiped (Figure 1). The origin of the coordinates is located in the lower left corner of a rectangular parallelepiped, as shown in the figure. The nodal points are numbered starting from the lower-left near corner (node 0). The dimensions of a rectangular parallelepiped along the x, y and z axes are considered equal to a, b and c, respectively. Convective heat exchange occurs on the face (0, 1, 2, 3), and a heat flow is applied to the face (4, 5, 6, 7).

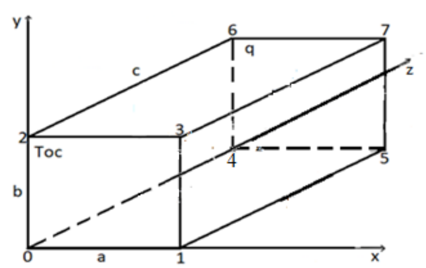

Figure 1. Diagram of the solid under study in the form of a rectangular parallelepiped

The task is to find the law of temperature distribution at any point of a solid body in the form of a rectangular parallelepiped.

Mathematically, the stationary problem is reduced to solving the heat equation:

$$\mathrm{K}xx(\frac{d^2T}{dx^2})+\mathrm{Kyy}(\frac{d^2T}{dy^2})+\mathrm{Kzz}(\frac{d^2T}{dz^2})=0, \qquad (1)$$

-under restrictions:
of the second kind

$$\mathrm{K}xx(\frac{d^2T}{dx^2})+\mathrm{Kyy}(\frac{d^2T}{dy^2})+\mathrm{Kzz}(\frac{d^2T}{dz^2})=0, \text{ on } S_1, \qquad (2)$$

- the third kind

$$(\mathrm{K}xx\frac{dT}{dx}+\mathrm{Kyy}\frac{dT}{dy}+\mathrm{Kzz}\frac{dT}{dz})\Big|_{s_2}+h(\mathrm{T}-Tat)=0, \text{on } S_2. \qquad (3)$$

Equation (2) is the boundary condition for heat flow, and equation (3) is for convective heat transfer.

The task is to find a solution to equation (1) under constraints (2) and (3) using a variational approach.

where T – temperature, ^0C;

q – heat flow, ^0C;

K_{xx}, K_{yy}, K_{zz} – thermal conductivity coefficients in coordinate directions *x, y* and *z,* $\frac{kBT}{m}$ ^0C;

h – heat transfer coefficient,

$$J_5=\frac{h}{2}\int_0^a\int_0^c(\mathrm{T}-\mathrm{T}_{at})^2dydz\Big|_{y=0},$$

$$J_6=\frac{h}{2}\int_0^a\int_0^c(\mathrm{T}-\mathrm{T}_{at})^2dxdz\Big|_{y=b},$$

$$J_7=\frac{h}{2}\int_0^a\int_0^c(\mathrm{T}-\mathrm{T}_{at})^2dydz\Big|_{x=a}, \qquad (4)$$

$$J_8=\frac{h}{2}\int_0^a\int_0^c(\mathrm{T}-\mathrm{T}_{at})^2dydz\Big|_{x=0},$$

S_1 – the surface on which the heat flow enters, m^2;

S_2 – the surface where heat exchange occurs, m^2;

T_{at} – ambient temperature, ^0C.

III. RESEARCH METHODOLOGY

To achieve this goal, the following tasks are solved:
1. Discretization of a rectangular parallelepiped along the x, y and z axes and construction of approximating spline functions for temperature within the length of each discrete element.
2. Formation of a functional of total thermal energy for all sampling elements, taking into account the boundary conditions when a heat flow enters one of the faces of a rectangular parallelepiped, and heat exchange with the environment occurs on the opposite side. And also,

consideration of cases when the remaining faces of a rectangular parallelepiped are thermally insulated and vice versa.

3. Minimization of the functional of the total thermal energy by the temperatures of the nodal points of a rectangular parallelepiped and obtaining resolving systems of linear algebraic equations to determine them.

4. Solving resolving systems of linear algebraic equations and obtaining the temperature value at the nodal points of a rectangular parallelepiped.

5. Determination of the temperature distribution law in a rectangular parallelepiped in accordance with the proposed temperature approximation formula.

6. Development of programs in Python for the formation of general functionality, solving resolving systems of linear equations and plotting the law of temperature distribution in a rectangular parallelepiped.

A variational approach is used to solve the problem [1]. According to this approach, the solution of the problem under consideration is equivalent to minimizing the temperature functional at the nodal points:

$$J=\int_V \frac{1}{2}[\mathrm{K}\,xx(\frac{dT}{dx})^2+\mathrm{Kyy}(\frac{dT}{dy})^2+\mathrm{Kzz}(\frac{dT}{dz})^2]dv+$$
$$\int_{S_1}\mathrm{q}\mathrm{dS}+\int_{S_2}\frac{h}{2}(\mathrm{T}-\mathrm{T}_{at})^2\,dS=J_1+J_2+J_3+J_4+J_5 \quad (5)$$

where V- the volume of the body in question.

For a rectangular parallelepiped (Figure 1), formula (7) has the form:

$$J=\frac{1}{2}\int_0^a\int_0^b\int_0^c[\mathrm{K}\,xx(\frac{dT}{dx})^2+\mathrm{Kyy}(\frac{dT}{dy})^2+\mathrm{Kzz}(\frac{dT}{dz})^2]+$$
$$\int_0^a\int_0^b qT\,dxdy\Big|_{z=c}+\frac{h}{2}\int_0^a\int_0^b(\mathrm{T}-Tat)^2\,dxdy\Big|_{z=0} \quad (6)$$

When the side faces of a rectangular parallelepiped are not thermally insulated, the following terms are added to the functional J, taking into account heat transfer:

$$J_5=\frac{h}{2}\int_0^a\int_0^c(\mathrm{T}-\mathrm{T}_{at})^2dydz\Big|_{y=0},$$
$$J_6=\frac{h}{2}\int_0^a\int_0^c(\mathrm{T}-\mathrm{T}_{at})^2dxdz\Big|_{y=b}, \quad (7)$$
$$J_7=\frac{h}{2}\int_0^a\int_0^c(\mathrm{T}-\mathrm{T}_{at})^2dydz\Big|_{x=a},$$
$$J_8=\frac{h}{2}\int_0^a\int_0^c(\mathrm{T}-\mathrm{T}_{at})^2dydz\Big|_{x=0},$$

where J_5, J_6, J_7, J_8 – characterize the heat exchange on the faces (0, 1, 4, 5), (2, 3, 6, 7), (0, 2, 4, 6), (1, 3, 5, 7) a rectangular parallelepiped, respectively.

In this case, the total functional is $J=\sum_{i=1}^{8}J_i$.

To minimize the functional J, the temperature T(x, y, z) is approximated by a third-order polynomial:

$$T(\mathrm{x,y,z})=\lambda_0+\lambda_1x+\lambda_2y+\lambda_3z+\lambda_4xy+\lambda_5xz+\lambda_6yz+\lambda_7xyz \quad (8)$$

Suppose that the temperature values are set at the nodal points of a rectangular parallelepiped:

$$\begin{aligned}
&T(\mathrm{x}_0,\mathrm{y}_0,\mathrm{z}_0)=\mathrm{T}_0,\\
&T(\mathrm{x}_1,\mathrm{y}_1,\mathrm{z}_1)=\mathrm{T}_1,\\
&T(\mathrm{x}_2,\mathrm{y}_2,\mathrm{z}_2)=\mathrm{T}_2,\\
&T(\mathrm{x}_3,\mathrm{y}_3,\mathrm{z}_3)=\mathrm{T}_3,\\
&T(\mathrm{x}_4,\mathrm{y}_4,\mathrm{z}_4)=\mathrm{T}_4,\\
&T(\mathrm{x}_5,\mathrm{y}_5,\mathrm{z}_5)=\mathrm{T}_5,\\
&T(\mathrm{x}_6,\mathrm{y}_6,\mathrm{z}_6)=\mathrm{T}_6,\\
&T(\mathrm{x}_7,\mathrm{y}_7,\mathrm{z}_7)=\mathrm{T}_7
\end{aligned} \quad (9)$$

Substituting expression (8) into the left part of these equations, we obtain a system of linear equations with respect to λ_0,λ_7:

$$\begin{cases}
\lambda_0+\lambda_1x_0+\lambda_2y_0+\lambda_3z_0+\lambda_4x_0y_0+\lambda_5x_0z_0+\lambda_6y_0z_0+\lambda_7x_0y_0z_0=T_0,\\
\lambda_0+\lambda_1x_1+\lambda_2y_1+\lambda_3z_1+\lambda_4x_1y_1+\lambda_5x_1z_1+\lambda_6y_1z_1+\lambda_7x_1y_1z_1=T_1,\\
\lambda_0+\lambda_1x_2+\lambda_2y_2+\lambda_3z_2+\lambda_4x_2y_2+\lambda_5x_2z_2+\lambda_6y_2z_2+\lambda_7x_2y_2z_2=T_2,\\
\lambda_0+\lambda_1x_3+\lambda_2y_3+\lambda_3z_3+\lambda_4x_3y_3+\lambda_5x_3z_3+\lambda_6y_3z_3+\lambda_7x_3y_3z_3=T_3,\\
\lambda_0+\lambda_1x_4+\lambda_2y_4+\lambda_3z_4+\lambda_4x_4y_4+\lambda_5x_4z_4+\lambda_6y_4z_4+\lambda_7x_4y_4z_4=T_4,\\
\lambda_0+\lambda_1x_5+\lambda_2y_5+\lambda_3z_5+\lambda_4x_5y_5+\lambda_5x_5z_5+\lambda_6y_5z_5+\lambda_7x_5y_5z_5=T_5,\\
\lambda_0+\lambda_1x_6+\lambda_2y_6+\lambda_3z_6+\lambda_4x_6y_6+\lambda_5x_6z_6+\lambda_6y_6z_6+\lambda_7x_6y_6z_6=T_6,\\
\lambda_0+\lambda_1x_7+\lambda_2y_7+\lambda_3z_7+\lambda_4x_7y_7+\lambda_5x_7z_7+\lambda_6y_7z_7+\lambda_7x_7y_7z_7=T_7
\end{cases} \quad (10)$$

There is a solution to this system:

$$\begin{cases}
\lambda_0=\mathrm{T}_0,\\
\lambda_1=\frac{-\mathrm{T}_0+\mathrm{T}_1}{\mathrm{a}},\\
\lambda_2=\frac{-\mathrm{T}_0+\mathrm{T}_2}{\mathrm{b}},\\
\lambda_3=\frac{-\mathrm{T}_0+\mathrm{T}_4}{\mathrm{c}},\\
\lambda_4=\frac{\mathrm{T}_0-\mathrm{T}_1-\mathrm{T}_2+\mathrm{T}_3}{\mathrm{ab}},\\
\lambda_5=\frac{\mathrm{T}_0-\mathrm{T}_1-\mathrm{T}_4+\mathrm{T}_5}{\mathrm{ac}},\\
\lambda_6=\frac{\mathrm{T}_0-\mathrm{T}_2-\mathrm{T}_4+\mathrm{T}_6}{\mathrm{bc}},\\
\lambda_7=\frac{-\mathrm{T}_0c+\mathrm{T}_1\mathrm{c}+\mathrm{T}_2c-\mathrm{T}_3c+\mathrm{T}_4\mathrm{c}-\mathrm{T}_5c-\mathrm{T}_6\mathrm{a}+\mathrm{T}_7a}{abc^2}
\end{cases} \quad (11)$$

After substituting these values into formula (5) and bringing similar terms, we get:

$$\begin{aligned}
&T(\mathrm{x,y,z})=\varphi_0(\mathrm{x,y,z})*\mathrm{T}_0+\varphi_1(\mathrm{x,y,z})*\mathrm{T}_1+\\
&\varphi_2(\mathrm{x,y,z})*\mathrm{T}_2+\varphi_3(\mathrm{x,y,z})*\mathrm{T}_3+\varphi_4(\mathrm{x,y,z})*\\
&\mathrm{T}_4+\varphi_5(\mathrm{x,y,z})*\mathrm{T}_5+\varphi_6(\mathrm{x,y,z})*\mathrm{T}_6+\varphi_7(\mathrm{x,y,z})*\mathrm{T}_7
\end{aligned} \quad (12)$$
$$-a\le x\le a;\ -b\le x\le b;\ -c\le x\le c,$$

Where

$$\varphi_0(\mathrm{x,y,z}) = 1 - \frac{z}{c} - \frac{y}{b} + \frac{yz}{bc} - \frac{x}{a} + \frac{xz}{ac} + \frac{xy}{ab} - \frac{xyz}{abc};$$
$$\varphi_1(\mathrm{x,y,z}) = \frac{x}{a} - \frac{xz}{ac} - \frac{xy}{ab} + \frac{xyz}{abc};$$
$$\varphi_2(\mathrm{x,y,z}) = \frac{y}{b} - \frac{yz}{bc} - \frac{xy}{ab} + \frac{xyz}{abc}; \qquad (13)$$
$$\varphi_3(\mathrm{x,y,z}) = \frac{xy}{ab} - \frac{xyz}{abc};$$
$$\varphi_4(\mathrm{x,y,z}) = \frac{z}{c} - \frac{yz}{bc} - \frac{xz}{bc} + \frac{xyz}{abc};$$
$$\varphi_5(\mathrm{x,y,z}) = \frac{xz}{ac} - \frac{xyz}{abc};$$
$$\varphi_6(\mathrm{x,y,z}) = \frac{yz}{bc} - \frac{xyz}{abc};$$
$$\varphi_7(\mathrm{x,y,z}) = \frac{xyz}{7abc};$$

Differentiating function (8) by variables x, y and z we get

$$\frac{\partial T}{\partial x} = \sum_{i=1}^{7} \frac{\partial \varphi_i}{\partial x} T_i;$$
$$\frac{\partial T}{\partial y} = \sum_{i=1}^{7} \frac{\partial \varphi_i}{\partial y} T_i;$$
$$\frac{\partial T}{\partial z} = \sum_{i=1}^{7} \frac{\partial \varphi_i}{\partial z} T_i;$$
$$\frac{\partial \varphi_0}{\partial x} = (-\frac{1}{a} + \frac{z}{ac} + \frac{y}{ab} - \frac{yz}{abc});$$
$$\frac{\partial \varphi_0}{\partial y} = (-\frac{1}{b} + \frac{z}{bc} + \frac{x}{ab} - \frac{xz}{abc});$$
$$\frac{\partial \varphi_7}{\partial x} = (\frac{yz}{abc});$$
$$\frac{\partial \varphi_7}{\partial y} = (\frac{xz}{abc}); \qquad (14)$$
$$\frac{\partial \varphi_7}{\partial z} = (\frac{xy}{abc});$$
$$\frac{\partial \varphi_1}{\partial x} = (\frac{1}{a} - \frac{z}{ac} - \frac{y}{ab} + \frac{yz}{abc});$$
$$\frac{\partial \varphi_1}{\partial y} = (-\frac{x}{ab} + \frac{xz}{abc});$$
$$\frac{\partial \varphi_1}{\partial z} = (-\frac{x}{ac} + \frac{xy}{abc});$$
$$\frac{\partial \varphi_2}{\partial x} = (-\frac{y}{ab} + \frac{yz}{abc});$$
$$\frac{\partial \varphi_2}{\partial y} = (\frac{1}{b} - \frac{z}{bc} - \frac{x}{ab} + \frac{xz}{abc});$$
$$\frac{\partial \varphi_2}{\partial z} = (-\frac{y}{ab} + \frac{xy}{abc});$$
$$\frac{\partial \varphi_3}{\partial x} = (\frac{y}{ab} + \frac{yz}{abc});$$
$$\frac{\partial \varphi_3}{\partial y} = (\frac{x}{ab} - \frac{xz}{abc});$$
$$\frac{\partial \varphi_3}{\partial z} = (-\frac{xy}{abc});$$
$$\frac{\partial \varphi_4}{\partial x} = (-\frac{z}{ac} + \frac{yz}{abc});$$
$$\frac{\partial \varphi_4}{\partial y} = (-\frac{z}{bc} + \frac{xz}{abc});$$
$$\frac{\partial \varphi_4}{\partial z} = (\frac{1}{c} - \frac{y}{bc} + \frac{xy}{abc});$$
$$\frac{\partial \varphi_5}{\partial x} = (\frac{z}{ac} - \frac{yz}{abc});$$
$$\frac{\partial \varphi_5}{\partial y} = (-\frac{xz}{abc});$$
$$\frac{\partial \varphi_5}{\partial z} = (\frac{x}{ac} - \frac{xy}{abc});$$
$$\frac{\partial \varphi_6}{\partial x} = (-\frac{yz}{abc});$$
$$\frac{\partial \varphi_6}{\partial y} = (\frac{y}{bc} - \frac{xz}{abc});$$
$$\frac{\partial \varphi_6}{\partial z} = (\frac{1}{bc} - \frac{xy}{abc});$$
$$\frac{\partial \varphi_7}{\partial x} = (\frac{yz}{abc});$$
$$\frac{\partial \varphi_7}{\partial y} = (\frac{xz}{abc});$$
$$\frac{\partial \varphi_7}{\partial z} = (\frac{xy}{abc});$$

Expression J after substitution of T and $(\frac{dT}{dx}, \frac{dT}{dy}, \frac{dT}{dz})$ it is calculated using the sympy module of the Python language, the expression of which is not given here because of the bulkiness.

To minimize the functional J , we differentiate it by variables $\overline{T_0, T_7}$ and we equate it to zero. As a result , we obtain a system of linear equations with respect to variables $\overline{T_0, T_7}$, which, for example, for the thermally insulated case, when we consider a cube with the length of the sides a, has the form:

$$
\begin{Bmatrix} d & 0 & 0 & f & 0 & f & f & f \\ 0 & d & f & 0 & f & 0 & f & f \\ 0 & f & d & 0 & f & f & 0 & f \\ f & 0 & 0 & d & f & f & f & 0 \\ 0 & f & f & f & d & 0 & 0 & f \\ f & 0 & f & f & 0 & d & f & 0 \\ f & f & 0 & f & 0 & f & d & 0 \\ f & f & f & 0 & f & 0 & 0 & d \end{Bmatrix} \begin{pmatrix} T_0 \\ T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \end{pmatrix} +
$$

$$
\begin{Bmatrix} g & \frac{g}{2} & \frac{g}{2} & \frac{g}{4} & 0 & 0 & 0 & 0 \\ \frac{g}{2} & g & \frac{g}{4} & \frac{g}{2} & 0 & 0 & 0 & 0 \\ \frac{g}{2} & \frac{g}{4} & g & \frac{g}{2} & 0 & 0 & 0 & 0 \\ \frac{g}{4} & \frac{g}{2} & \frac{g}{2} & g & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{Bmatrix} \begin{pmatrix} T_0 \\ T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \end{pmatrix} = \begin{pmatrix} l \\ l \\ l \\ l \\ p \\ p \\ p \\ p \end{pmatrix}, \qquad (15)
$$

$$
l = Tat * a ** 2 * h / 4
$$
$$
p = -a ** 2 * q / 4
$$

Here, it should be noted that the developed Python program allows you to obtain a system of linear equations for any number of partitions of the sides of a rectangular parallelepiped.

The solution of the resulting system of equations makes it possible to determine the temperatures at the nodal points of a rectangular parallelepiped. Substituting these values in (8), we obtain the law of temperature distribution at any point of a rectangular parallelepiped.

All these operations were performed using the sympy module of the Python language.

IV. Practical Implementation

For the practical implementation of the proposed approach, a Python program was developed. As an example, a cube was selected (Figure 2) with the following initial data: $a = 0.015m,\ b = 0.015m,\ c = 0.015m,$

$$
K_{xx} = 75000\frac{BT}{m^{\circ}C}, T_{at} = 40^{\circ}C, q = -500\frac{kBT}{m^{2\,0}C}
$$

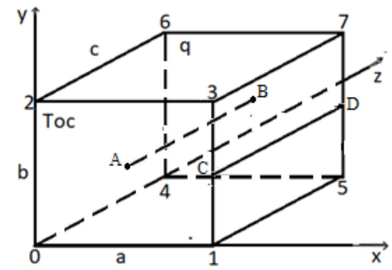

Figure 2. A solid body in the shape of a cube, consisting of a single element

If we denote the number of partitions of a rectangular parallelepiped into elements along the x, y and z axes as mx, my and mz, respectively, then for a cube we have mx=my=mz=m.

Let's introduce arrays of temperatures *T1[0,8]*, *T2[0,27]*, *T3[0,63]*, *T4[0,124]*, corresponding to the nodal points of the cube for the number partitions *m=1,2,3,4*, accordingly. The temperature values at the nodal points of the cube for the thermally insulated case turned out to be equal

1) when *m =1* (Figure 2):
   T1[0,3]=90, *T1[4,7]=90;*
2) when *m =2* (Figure 3):
   T2[0,8]=90, *T2[9,17]=95*, *T2[18,27]=100 ;*

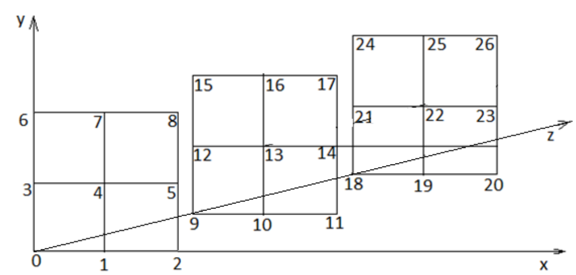

Figure 3. A solid body in the shape of a cube when divided into 8 elements

3) when *m =3* (Figure 4):
   T3[0,15]=90, *T3[16,31]=93.3*, *T3[32,47]=96.6*, *T4[48,63]=100;*

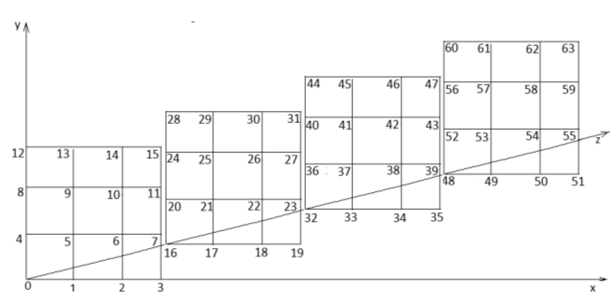

Figure 4. A solid body in the shape of a cube when divided into 27 elements

4) when *m =4* (Figure 5):
   T4[0,24]=90, *T4[25,49]=92.5*, *T4[50,74]=95*, *T4[75,99]=97.5*, *T4[99,124]=100;*

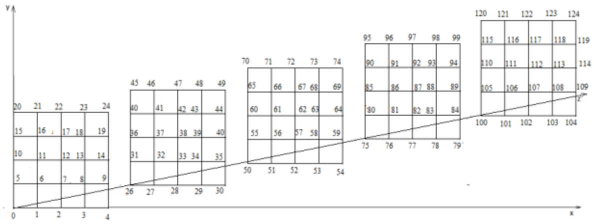

Figure 5. A solid body in the form of a cube when divided into 125 elements

The laws of temperature distribution for the thermally insulated case, when *m=1, m=2, m=3* and *m=4* they turned out to be the same, corresponding to a straight line between the points (0, 90) and (0.015, 100) (Figure 6).

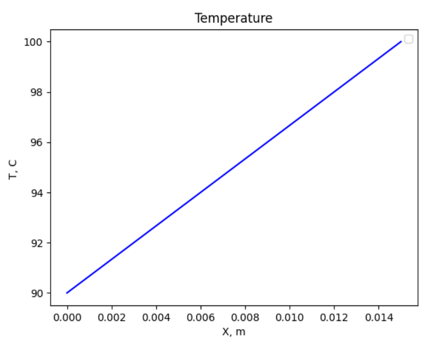

Figure 6. The law of temperature distribution in a cube for the thermally insulated case at *m=1*, *m=2*, *m=3* and *m=4*

From the data *T4[0,124]* it can be seen that in the thermally insulated case, the temperature in the sections of the cube perpendicular to the z axis is the same.

The temperature values at the nodal points of the cube for the non-insulated case when *m =1*, *m =2*, *m =2* and *m=4* turned out to be equal:

1) *m =1* (Figure 2):

T1[0,7]= [47.831326, 47.831326, 47.831326, 47.831326, 53.253014, 53.253014, 53.253014, 53.253014]

2) *m =2* (Figure 3):

*T2[0,27]= [48.013096, 48.41593, 48.013096,
48.41593, 48.839104, 48.41593,
48.013096, 48.41593, 48.013096,*

*49.642757, 50.13417, 49.642757,
50.13417, 50.650364, 50.13417,
49.642757, 50.13417, 49.642757,*

*53.277725, 53.897366, 53.277725,
53.897366, 54.547207, 53.897366,
53.277725, 53.897366, 53.277725]*

3) *m =3* (Figure 4):

*T3[0,63]= [48.041508, 48.4015, 48.4015, 48.041508,
48.4015, 48.777603, 48.777603, 48.4015,
48.4015, 48.777603, 48.777603, 48.4015,
48.041508, 48.4015, 48.4015, 48.041508,*

*48.937553, 49.337643, 49.337643, 48.937553,
49.337643, 49.755657, 49.755657, 49.337643,
49.337643, 49.755657, 49.755657, 49.337643,
48.937553, 49.337643, 49.337643, 48.937553,*

*50.61583, 51.09144, 51.09144, 50.61583,
51.09144, 51.588215, 51.588215, 51.09144,
51.09144, 51.588215, 51.588215, 51.09144,
50.61583, 51.09144, 51.09144, 50.61583,*

*53.286686, 53.833485, 53.833485, 53.286686,
53.833485, 54.403946, 54.403946, 53.833485,
53.833485, 54.403946, 54.403946, 53.833485,
53.286686, 53.833485, 53.833485, 53.286686].*

5) *m =4* (Figure 5):

*T4[0,124]= [48.05157,
48.355335,48.457405,48.355335,48.05157,
48.355335,48.67056, 48.776478,48.67056,
48.355335,
48.457405,48.776478,48.88369,
48.776478,48.457405,
48.355335,48.67056, 48.776478,48.67056,
48.355335,
48.05157,
48.355335,48.457405,48.355335,48.05157,*

*48.65424,
48.980705,49.090366,48.980705,48.65424,
48.980705,49.31949, 49.433285,49.31949,
48.980705,*

*49.090366,49.433285,49.548473,49.433285,49.0903
66,
48.980705,49.31949, 49.433285,49.31949,
48.980705,
48.65424,
48.980705,49.090366,48.980705,48.65424,*

*49.680595,50.04532, 50.16767, 50.04532,
49.680595,
50.04532, 50.42379, 50.55075, 50.42379,
50.04532,
50.16767, 50.55075, 50.679253,50.55075,
50.16767,
50.04532, 50.42379, 50.55075, 50.42379,
50.04532,
49.680595,50.04532, 50.16767, 50.04532,
49.680595,*

*51.184772,51.603752,51.74121,
51.603752,51.184772,*

51.603752,52.038383,52.180992,52.038383,51.603752,
51.74121, 52.180992,52.325314,52.180992,51.74121,

51.603752,52.038383,52.180992,52.038383,51.603752,
51.184772,51.603752,51.74121, 51.603752,51.184772,

53.28972, 53.75355, 53.896935,53.75355, 53.28972,
53.75355, 54.2343, 54.383083,54.2343, 53.75355,
53.896935,54.383083,54.53367, 54.383083,53.896935,
53.75355, 54.2343, 54.383083,54.2343, 53.75355,
53.28972, 53.75355, 53.896935,53.75355, 53.28972].

Consider the temperatures for the edge of the cube passing through the nodes:
- (0-4) at *m =1*. Here the temperature in the nodes is *T1= [47.831326, 53.253014];*
- (0,9,18) at *m =2*. Here the temperature in the nodes is *T2= [48.013096, 49.642757, 53.277725];*
- (0,16,32,48) at *m* =3. Here the temperature in the nodes is *T3= [48.041508, 48.937553, 50.61583, 53.286686]*.
- (0,26,50,75,100) at *m =4*. Here the temperature in the nodes is *T4= [48.05157, 48.65424, 49.680595, 51.184772653.28972]*.

The temperature distribution laws for these nodes are shown in Figure 7.

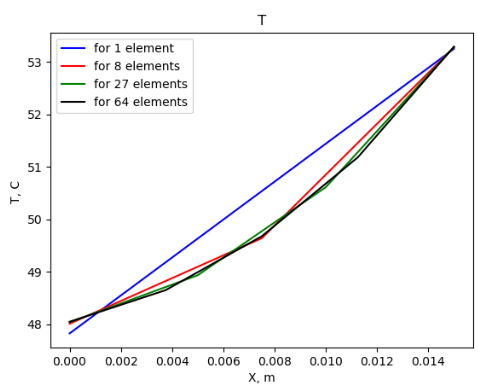

Figure 7. The law of temperature distribution in a cube for the non-insulated case at *m=1*, *m=2*, *m=3* and *m=4*

Comparison of the temperature distribution laws for a rectangular parallelepiped (mx=my=1 and mz=5) and a rod (m=5) under the influence of temperature and the presence of heat exchange in the non-insulated case

The figure in blue shows the laws of temperature distribution for the thermally insulated case for *m =1*, red for *m=2*, green for *m* =3, and black for m =4.

Figure 7 also shows that the temperature distribution law for the non-insulated case is nonlinear.

To find the maximum relative error, we find the deviation of the graphs for the case *m =2* and *m =3*. Consider the values of these graphs in points *x=(0,0.005, 0.0075,0.01,0.015)*, where linearity changes (table 1).

Table 1. Temperature deviation for *m =2* and *m =3*

X	*0*	*0.005*	*0.0075*	*0.01*	*0.015*
T2	*48.01*	*49.1*	*49.6*	*50.85*	*53.28*
T3	*48.04*	*49.1*	*49.6*	*50.85*	*53.28*
*D=abs(*T3-T3*)*	*0.03*	*0.17*	*0.18*	*0.23*	*0.01*

Find the maximum value in the table row *d: dmax=0.23* and the minimum value from the string *m* =3: *Tmin=48.04*.

The maximum relative error in percent does not exceed: $\frac{d\max}{T\min}100\% = \frac{23}{48.04} = 0.5\%$

When comparing graphs for *m=3* and *m=4* according to the above method, the relative error does not exceed *d= 0.2%*.

This suggests that to obtain a relative error not exceeding 0.5%, splitting the cube along the x, y and z axes into 3 elements is sufficient, and for this error not to exceed 0.2%, splitting the cube into 4 elements is sufficient.

Let us now compare the laws of temperature distribution along some lines of the cube parallel to the axis *z* for *m=3*.

Temperatures at *m =3* for the edge of the cube (nodes 0, 16, 32,48) are equal to *Tr= [48.041508, 48.937553, 50.61583, 53.286686]*.

The values of Tc temperatures at the nodes along the line passing through the center of the cube (line AB of Figure 2) parallel to the z axis are determined by (6):

$$T_c[0] = \varphi_0(a/6, a/6, 0)T[5] + \varphi_0(a/6, a/6, 0)T[6] + \varphi_0(a/6, a/6, 0)T[9] + \varphi_0(a/6, a/6, 0)T[10] + \varphi_0(a/6, a/6, 0)T[21] + \varphi_0(a/6, a/6, 0)T[22] + \varphi_0(a/6, a/6, 0)T[25] + \varphi_0(a/6, a/6, 0)T[26];$$

$$T_c[1] = \varphi_0(a/6, a/6, a/3)T[5] + \varphi_0(a/6, a/6, a/3)T[6] + \varphi_0(a/6, a/6, a/3)T[9] + \varphi_0(a/6, a/6, a/3)T[10] + \varphi_0(a/6, a/6, a/3)T[21] + \varphi_0(a/6, a/6, a/3)T[22] + \varphi_0(a/6, a/6, a/3)T[25] + \varphi_0(a/6, a/6, a/3)T[26];$$

$$T_c[2] = \varphi_0(a/3, a/6, 0)T[37] + \varphi_0(a/3, a/6, 0)T[38] + \varphi_0(a/3, a/6, 0)T[41] + \varphi_0(a/3, a/6, 0)T[42] + \varphi_0(a/3, a/6, 0)T[53] + \varphi_0(a/3, a/6, 0)T[54] + \varphi_0(a/3, a/6, 0)T[57] + \varphi_0(a/3, a/6, 0)T[58];$$

$$T_c[3]=\varphi_0(a/3,a/6,a/3)T[38]+\varphi_0(a/3,a/6,a/3)T[39]+$$
$$\varphi_0(a/3,a/6,a/3)T[42]+\varphi_0(a/3,a/6,a/3)T[43]+$$
$$\varphi_0(a/3,a/6,a/3)T[54]+\varphi_0(a/3,a/6,a/3)T[55]+$$
$$\varphi_0(a/3,a/6,a/3)T[58]+\varphi_0(a/3,a/6,a/3)T[59];$$

As a result, we get *Tc= [48.777603 49.755657 51.588215 54.1187155]*.

The temperature values Tr for the middle of the right face of the cube (line CD of Figure 2) are also obtained by (6):

$$T_r[0]=\varphi_0(a/3,a/6,0)T[6]+\varphi_0(a/3,a/6,0)T[7]+$$
$$\varphi_0(a/3,a/6,0)T[9]+\varphi_0(a/3,a/6,0)T[10]+$$
$$\varphi_0(a/3,a/6,0)T[22]+\varphi_0(a/3,a/6,0)T[23]+$$
$$\varphi_0(a/3,a/6,0)T[26]+\varphi_0(a/3,a/6,0)T[27];$$

$$T_r[1]=\varphi_0(a/3,a/6,c/3)T[6]+\varphi_0(a/3,a/6,c/3)T[7]+$$
$$\varphi_0(a/3,a/6,3/2)T[9]+\varphi_0(a/3,a/6,c/3)T[10]+$$
$$\varphi_0(a/3,a/6,c/3)T[22]+\varphi_0(a/3,a/6,c/3)T[23]+$$
$$\varphi_0(a/3,a/6,c/3)T[26]+\varphi_0(a/3,a/6,c/3)T[27];$$

$$T_r[2]=\varphi_0(a/3,a/6,0)T[38]+\varphi_0(a/3,a/6,0)T[39]+$$
$$\varphi_0(a/3,a/6,0)T[42]+\varphi_0(a/3,a/6,0)T[43]+$$
$$\varphi_0(a/3,a/6,0)T[54]+\varphi_0(a/3,a/6,0)T[55]+$$
$$\varphi_0(a/3,a/6,0)T[58]+\varphi_0(a/3,a/6,0)T[59];$$

$$T_r[3]=\varphi_0(a/3,a/6,c/3)T[38]+\varphi_0(a/3,a/6,c/3)T[39]+$$
$$\varphi_0(a/3,a/6,c/3)T[42]+\varphi_0(a/3,a/6,c/3)T[43]+$$
$$\varphi_0(a/3,a/6,c/3)T[54]+\varphi_0(a/3,a/6,c/3)T[55]+$$
$$\varphi_0(a/3,a/6,c/3)T[58]+\varphi_0(a/3,a/6,c/3)T[59];$$

As a result , we get *Tr= [47.83, 48.41593, 50.13417, 53.897366]*.

The laws of temperature distribution for these cases are shown in Figure 8.

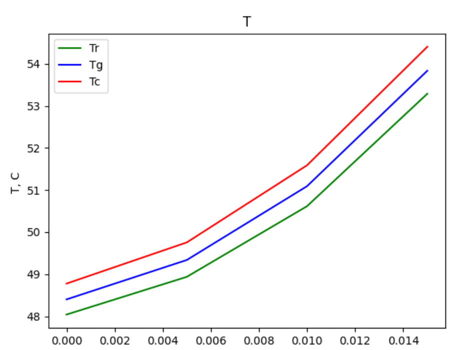

Figure 8. Temperature distribution laws for the edge, the middle of the face and the center of the cube

It can be seen from the figure that the temperature values along the line AB of Figure 2 passing through the center of the cube are greater than the temperature on the edge (0-4) of Figure 2 and the temperature on the line passing through the middle of the cube face (line CD of Figure 2). In turn, the temperature on the CD line of Figure 2 is greater than the temperature on the AB line of Figure 2. This means that the farther the line is from the center, the lower its temperature.

In order to verify the correctness of the proposed approach and the calculations performed, a comparative analysis of the temperature distribution laws for a rectangular parallelepiped was carried out at *mx=my=1* and *mz=5* (figure 9), and the rod (figure 10) with the number of elements *m=5*, considered in [23].

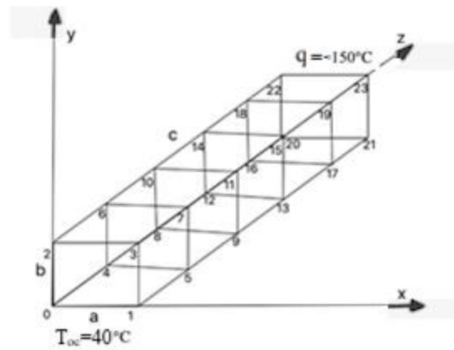

Figure 9. A solid body in the form of a rectangular parallelepiped at *mx=my=1* and *mz=5*.

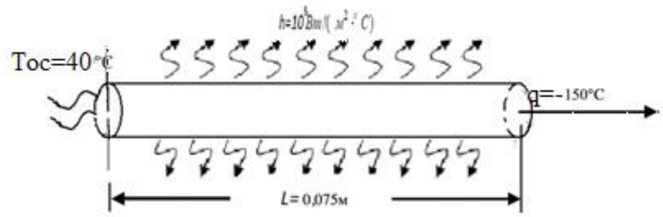

Figure 10. Design scheme of the rod.

The initial data for a rectangular parallelepiped were taken equal to:

$$a=0.015m, b=0.015m, c=0.075, K_{xx}=7500\frac{BT}{m^0C},$$
$$T_{at}=40^0C, q=-150^0C, h=100000\frac{BT}{m^{2\,0}C}.$$

Temperature values at the nodal points along the axis *z* rectangular parallelepiped (when *mx=my=1*, *mz=5*) (when exposed to temperature at the border (20-21-22-23), and at the nodal points of the rod (at m= 5) for the non-insulated case are presented in Tables 2 and 3, and the laws of temperature distribution in Figure 11. The calculation results for the rod were obtained according to the methodology described in [23].

Table 2. Non-insulated case - parallelepiped

x	0	0.015	0.03	0.045	0.06	0.075
T	40.05	40.09	40.2	40.5	41.3	43.2

Table 3. Non-insulated case – rod

X	0	0.015	0.03	0.045	0.06	0.075
T	40.1	40.2	40.4	40.8	41.7	43.8

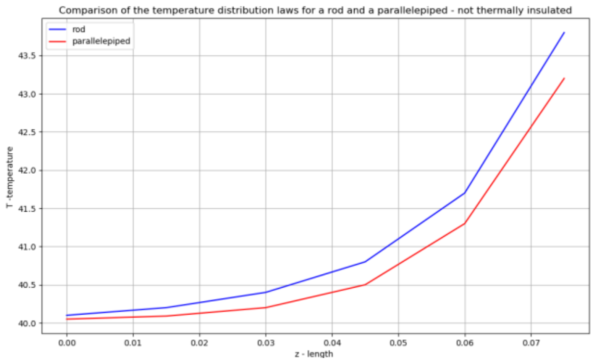

Figure 11. Comparison of temperature distribution laws for a rectangular parallelepiped (*mx=my=1 и mz=5*) and the rod (m=5) under the influence of temperature and the presence of heat exchange in the non-insulated case

Figure 10 shows in red the temperature distribution law for a rectangular parallelepiped, and in blue for a rod.

The figure also shows that the temperature distribution laws for a rectangular parallelepiped and a rod are very close, and from Tables 2 and 3 it can be determined that the maximum relative error does not exceed:

$$(43.8-43.2)*\frac{100}{43.2}=1.39\%.$$

This difference is due to the different shape and cross-sectional area of a rectangular parallelepiped (0.0004 m^2) and the rod (0.000314 m^2).

The temperature values at the nodal points along the z axis of the rectangular parallelepiped and the rod for the thermally insulated case with the same initial data completely coincided, and are presented in Table 4.

Table 4. Thermally insulated case - rectangular parallelepiped and rod

x	0	0.015	0.03	0.045	0.06	0.075
T	55	58	61	64	67	70

The proximity of the results of comparing the temperature distribution laws for a rectangular parallelepiped and a rod shows the possibility of using the proposed approach to solve practical problems of heat transfer.

V. Conclusion

In this paper, a general variational functional is obtained for determining the law of temperature distribution in the body of a rectangular parallelepiped shape, when a heat flow enters one of the faces of a rectangular parallelepiped, and heat exchange with the environment occurs on the opposite side. Nonlinear temperature approximation is used to minimize the obtained functional at discrete points. The minimization of the general functional by the temperatures set at the nodal points is carried out. At the same time, the minimization problem is reduced to solving systems of linear equations. The temperature values at the nodal points are used to estimate the temperature distribution law at any point of the body in the form of a rectangular parallelepiped. The proposed approach is applied to solve specific problems when a heat flow is applied to one side of the parallelepiped, and heat exchange with the environment occurs on the opposite side. It is shown that even when dividing a rectangular parallelepiped into three elements, the relative temperature error does not exceed 0.5%.

To test the proposed approach, a comparative analysis of the laws of temperature distribution in the body of a rectangular parallelepiped shape, whose length in z is much greater than the length of the other sides, with a rod with similar geometric characteristics, is carried out. It turned out that the relative error in determining temperatures does not exceed 1.39%. This error is due to the difference in the shape and cross-sectional area of the rod and the rectangular parallelepiped perpendicular to the z axis.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

-Kazykhan Rysgul carried out the discretization of a rectangular parallelepiped along the x, y and z axes and construction of approximating spline functions for temperature within the length of each discrete element.

-Tashev Azat has implemented the formation of a functional of total thermal energy for all sampling elements, taking into account the boundary conditions when a heat flow enters one of the faces of a rectangular parallelepiped, and heat exchange with the environment occurs on the opposite side. And also, consideration of cases when the remaining faces of a rectangular parallelepiped are thermally insulated and vice versa.

-Aitbayeva Rakhatay has implemented the minimization of the functional of the total thermal energy by the temperatures of the nodal points of a rectangular parallelepiped and obtaining resolving systems of linear algebraic equations to determine them.

-Kudaykulov Anarbay was responsible for the solving resolving systems of linear algebraic equations and obtaining the temperature value at the nodal points of a rectangular parallelepiped.

-Kunelbayev Murat solution determination of the temperature distribution law in a rectangular parallelepiped in accordance with the proposed temperature approximation formula.

-Kuanysh Dauren solution development of programs in Python for the formation of general functionality, solving resolving systems of linear equations and plotting the law of temperature distribution in a rectangular parallelepiped.

-Mukaddas Arshidinova has organized and executed the experiments of Section 4