Approximation of a wanted flow via topological sensitivity analysis

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Abstract—We propose an optimization algorithm for the geometric control of fluid flow. the used approach is based on the topological sensitivity analysis method. It consists in studying the variation of a cost function with respect to the insertion of a small obstacle in the domain. Some theoretical and numerical results are presented in 2D and 3D.

Keywords—Sensitivity analysis, topological gradient, shape optimization, Stokes equations.

I. Introduction

THE optimal control of fluid flows has long been receiving considerable attention by engineers and mathematicians due to its importance in many applications involving fluid related technology [11], [16]. There is a wealth of literature on optimal control of flows through suction and injection of fluid along domain boundaries, see e.g. [7], [12]. In the context of design, one of the first studies is found in [18]. It is devoted to determine a minimum drag profile submerged in a homogeneous, steady, viscous fluid by using optimal control theories for distributed parameter systems. Next, many shape optimization methods are introduced to determine the design of minimum drag bodies [8], [15], [19], diffusers [5], and airfoils [6], [17]. The majority of works dealing with optimal design of flow domains fall into the category of shape optimization and are limited to determine the optimal shape of an existing boundary.

It is only recently that topological optimization has been developed and used in fluid design problems. It can be used to design features within the domain allowing new boundaries to be introduced into the design. In this context, one of the first approaches is proposed by Borvall and Petersson in [3]. They implemented the relaxed material distribution approach to minimize the power dissipated in Stokes flow. To approximate the no-slip condition along the solid-fluid interface they used a generalized Stokes problem to model fluid flow throughout the domain. Later, this approach has been generalized by Guest and Prévast in [9]. They treated the material phase as a porous medium where fluid flow is governed by Darcy's law. For impermeable solid material, the no-slip condition is simulated by using a small value for the material permeability to obtain negligible fluid velocities at the nodes of solid elements. The flow regularization is expressed as a system of equations; Stokes flow governs in void elements and Darcy flow governs in solid elements.

In this paper, we propose a new, fast and accurate optimization algorithm based on topological sensitivity analysis [1], [2], [10], [13], [14], [20]. It consists in studying the variation of a cost function with respect to a small topological perturbation of the fluid flow domain.

To present the basic idea, let us consider a domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ and a cost function $j(\Omega) = J(\Omega, u_\Omega)$, where u_Ω is the velocity field solution to Stokes problem defined in Ω. For $\varepsilon > 0$, let $\Omega_\varepsilon = \Omega \backslash \overline{(x_0 + \varepsilon\omega)}$ be the fluid domain obtained by inserting a small obstacle $x_0 + \varepsilon\omega$ in Ω, where $x_0 \in \Omega$ and $\omega \subset \mathbb{R}^d$ is a fixed bounded domain containing the origin, whose boundary $\partial\omega$ is connected and piecewise of class $\mathcal{C}^1$. The topological sensitivity analysis method leads to an asymptotic expansion of the function j in the following form:

$$j(\Omega_\varepsilon) = j(\Omega) + f(\varepsilon)g(x_0) + o(f(\varepsilon)),$$

where $f(\varepsilon)$ is a scalar positive function going to zero with ε. This expression is called the topological asymptotic expansion and g is called the topological gradient. The function g is very easy to compute. In order to minimize the cost function, the best location to insert a small obstacle in Ω is where g is negative. In fact if $g(x_0) < 0$, we have $j(\Omega_\varepsilon) < j(\Omega)$ for small ε. Starting with this observation, a topological optimization algorithm can then be constructed. The optimal design is obtained using an iterative process building a sequence of geometries $(\Omega_k)_k$ with $\Omega_0 = \Omega$. At the k^{th} iteration the topological gradient g_k is computed in Ω_k and the new geometry Ω_{k+1} is obtained by inserting an obstable ω_k in the domain

Ω_k; $\Omega_{k+1} = \Omega_k \backslash \overline{\omega_k}$. The obstacle ω_k is defined by a level set curve of g_k

$$\omega_k = \{x \in \Omega_k, \text{ such that } g_k(x) \leq c_k < 0\},$$

where c_k is chosen in such a way that the cost function j decreases as most as possible. This algorithm can be seen as a descent method where the descent direction is determined by the topological sensitivity g_k and the step length is given by the volume variation $meas(\Omega_k \backslash \Omega_{k+1})$.

The paper is organized as follows. In section 2, we give a statement of the optimization problem. Section 3 is devoted to a topological sensitivity analysis for the Stokes equations. The obtained results are valid for a large class of cost functions. Similar analysis is developed by Guillaume and SidIdris in [10]. Their approach is based on an adaptation of the adjoint method and a domain truncation technique that provides an equivalent formulation of the PDE in a fixed functional space. In this work, we derive a simplified topological sensitivity analysis for the Stokes equations without using the truncation technique. In section 4, we present some numerical experiments showing the efficiency of our approach.

II. Topological optimization problem

Consider a viscous incompressible fluid flow in a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. We assume that the fluid flow is governed by the Stokes equations.
We denote by $\Omega \backslash \overline{\omega_\varepsilon}$ the perturbed domain, obtained by inserting a small obstacle $\omega_\varepsilon = x_0 + \varepsilon\omega$ in the initial domain flow Ω. In $\Omega \backslash \overline{\omega_\varepsilon}$, the velocity u_ε and the pressure p_ε are solution to

$$\left\{ \begin{array}{rll} -\nu\Delta u_\varepsilon + \nabla p_\varepsilon & = F & \text{in } \Omega \backslash \overline{\omega_\varepsilon} \\ \text{div } u_\varepsilon & = 0 & \text{in } \Omega \backslash \overline{\omega_\varepsilon} \\ u_\varepsilon & = 0 & \text{on } \Gamma \\ u_\varepsilon & = 0 & \text{on } \partial\omega_\varepsilon. \end{array} \right. \quad (1)$$

where ν is the (constant) fluid kinematic viscosity, and F is a given body force per unit of mass. Note that for $\varepsilon = 0$, $(u_0\,,\, p_0)$ is solution to

$$\left\{ \begin{array}{rll} -\nu\Delta u_0 + \nabla p_0 & = F & \text{in } \Omega \\ \text{div } u_0 & = 0 & \text{in } \Omega \\ u_0 & = 0 & \text{on } \Gamma. \end{array} \right. \quad (2)$$

Consider now a design function j of the form

$$j(\Omega \backslash \overline{\omega_\varepsilon}) = J_\varepsilon(u_\varepsilon), \quad (3)$$

where J_ε is defined on $H^1(\Omega \backslash \overline{\omega_\varepsilon})^d$ for $\varepsilon \geq 0$
Our aim is to determine the optimal location of the obstacle ω_ε in the domain Ω in order to minimize the cost function $J_\varepsilon(u_\varepsilon)$. Then, the optimization problem we consider is given as follows:

$$\begin{array}{l} \min\limits_{\omega_\varepsilon \subset \Omega} J_\varepsilon(u_\varepsilon) \text{ such that, for some } p_\varepsilon, \\ (u_\varepsilon, p_\varepsilon) \text{ is a solution of (1) in } \Omega \backslash \overline{\omega_\varepsilon}. \end{array} \quad (4)$$

To this end, we will derive a topological asymptotic expansion of the function j with respect to ε.

III. Topological sensitivity analysis

In our topological sensitivity analysis, we have to distinguish the cases $d = 2$ and $d = 3$. This is due to the fact that the fundamental solutions $(E,\, \Pi)$ to the Stokes equations in $\mathbb{R}^2$ and $\mathbb{R}^3$ have essentially different asymptotic behaviour at infinity. We have if $d = 3$

$$E(y) = \frac{1}{8\pi\nu r}\Big(I + e_r e_r^T\Big),$$
$$\Pi(y) = \frac{y}{4\pi r^3}$$

and if $d = 2$

$$E(y) = \frac{1}{4\pi\nu}\Big(-\log(r)I + e_r e_r^T\Big),$$
$$\Pi(y) = \frac{y}{2\pi r^2},$$

with $r = ||y||$, $e_r = y/r$ and e_r^T is the transposed vector of e_r.

Next we assume that J_ε satisfies the following assumption.

Hypothesis 3.1: i) J_0 is differentiable with respect to u, its derivative being denoted by $DJ_0(u)$.
ii) There exists a real number δJ such that $\forall\, \varepsilon \geq 0$

$$\begin{array}{rl} J_\varepsilon(u_\varepsilon) - J_0(u_0) = & DJ_0(u_0)(\widehat{u}_\varepsilon - u_0) \\ & +f(\varepsilon)\delta J + o(\varepsilon), \end{array} \quad (5)$$

where f is a scalar function and $\widehat{u}_\varepsilon$ is an extension of u_ε in Ω respectively defined by:

$$f(\varepsilon) = \left\{ \begin{array}{ll} \varepsilon & \text{if } d = 3, \\ -1/\log(\varepsilon) & \text{if } d = 2, \end{array} \right.$$

$$\widehat{u}_\varepsilon = \left\{ \begin{array}{ll} u_\varepsilon & \text{in } \Omega \backslash \overline{\omega_\varepsilon}, \\ 0 & \text{in } \omega_\varepsilon. \end{array} \right.$$

A- The three dimensional case: Let $(U,\, P)$ denotes a solution to

$$\left\{ \begin{array}{rll} -\nu\Delta U + \nabla P & = 0 & \text{in } \mathbb{R}^3 \backslash \overline{\omega} \\ \text{div } U & = 0 & \text{in } \mathbb{R}^3 \backslash \overline{\omega} \\ U & \longrightarrow 0 & \text{at } \infty \\ U & = -u_0(x_0) & \text{on } \partial\omega. \end{array} \right. \quad (6)$$

We start the derivation of the topological asymptotic expansion with the following estimate of the

$H^1(\Omega\backslash\overline{\omega_\varepsilon})$ norm of $u_\varepsilon(x)-u_0(x)-U(x/\varepsilon)$. This estimate plays a crucial role in the derivation of our topological asymptotic expansion. It describes the velocity perturbation caused by the presence of the small obstacle ω_ε.

Proposition 3.1: There exists $c>0$, independent of ε, such that for all $\varepsilon>0$ we have

$$\|u_\varepsilon(x)-u_0(x)-U(x/\varepsilon)\|_{1,\Omega\backslash\overline{\omega_\varepsilon}}\leq c\,\varepsilon.$$

The following corollary follows from Proposition 3.1. It gives the behaviour of the velocity u_ε when inserting an obstacle. The principal term of this perturbation is given by the function U, solution to (6).

Corollary 3.1: We have

$$u_\varepsilon(x)=u_0(x)+U(x/\varepsilon)+O(\varepsilon),\quad x\in\Omega\backslash\overline{\omega_\varepsilon}.$$

We are now ready to derive the topological asymptotic expansion of the cost function j. It consists in computing the variation $j(\Omega\backslash\overline{\omega_\varepsilon})-j(\Omega)$ when inserting a small obstacle inside the domain. The leading term of this variation involves the solution to a boundary integral equation (see Theorem 3.1).

Theorem 3.1: [13] If the assumption 3.1 holds, the function j has the following asymptotic expansion

$$j(\Omega\backslash\overline{\omega_\varepsilon})=\quad j(\Omega)+\varepsilon\left[\left(-\int_{\partial\omega}\eta(y)\ \mathrm{ds(y)}\right).v_0(x_0)+\delta J\right]+o(\varepsilon),$$

where v_0 is the solution to the adjoint problem

$$\left\{\begin{array}{rll}-\nu\Delta v_0+\nabla q_0 &=-DJ(u_0) & \text{in }\Omega\\ \text{div }v_0 &=0 & \text{in }\Omega\\ v_0 &=0 & \text{on }\Gamma.\end{array}\right.$$

The function $\eta\in H^{-1/2}(\partial\omega)^3$ is the solution to the following boundary integral equation

$$\int_{\partial\omega}E(y-x)\,\eta(x)\ \mathrm{ds}(x)=-u_0(x_0),\quad\forall y\in\partial\omega.$$

In the particular case where $\omega=B(0,1)$, the density η is given explicitly $\eta(y)=-\dfrac{3\nu}{2}u_0(x_0),\quad\forall y\in\partial\omega.$

Corollary 3.2: If $\omega=B(0,1)$, under the assumption 3.1 we have

$$j(\Omega\backslash\overline{\omega_\varepsilon})=\quad j(\Omega)+\varepsilon\left[6\pi\nu\,u_0(x_0).v_0(x_0)+\delta J\right]+o(\varepsilon).$$

B- The two dimensional case: In the two dimensional case we have the following asymptotic expansion.

Theorem 3.2: If the assumption 3.1 holds, j admits the following asymptotic expansion

$$j(\Omega\backslash\overline{\omega_\varepsilon})=j(\Omega)+\quad\tfrac{-1}{\log(\varepsilon)}\left[4\pi\nu\,u_0(x_0).v_0(x_0)+\delta J\right]+o\left(\tfrac{-1}{\log(\varepsilon)}\right).$$

IV. Numerical examples

We consider a tank Ω filled with a viscous and incompressible fluid. The aim is to determine the optimal shape of the fluid flow domain minimizing a given objective function.

Our implementation is based on the following optimization algorithm. We apply an iterative process to build a sequence of geometries $(\Omega_k)_{k\geq0}$ with $\Omega_0=\Omega$. At the k^{th} iteration the topological gradient g_k is computed in Ω_k and the new geometry Ω_{k+1} is obtained by inserting an obstacle ω_k in the domain Ω_k; $\Omega_{k+1}=\Omega_k\backslash\overline{\omega_k}$. The obstacle ω_k is defined by a level set curve of g_k

$$\omega_k=\{x\in\Omega_k,\ \text{such that }g_k(x)\leq c_k<0\},$$

where c_k is chosen in such a way that the cost function j decreases as much as possible.

The algorithm :

- Initialization: choose $\Omega_0=\Omega$, and set $k=0$.
- Repeat until $g_k\geq0$ in Ω_k:
  - solve the Stokes equations in Ω_k,
  - solve the associated adjoint problem in Ω_k,
  - compute the topological sensitivity $g_k(x)$ $\forall x\in\Omega_k$,
  - determine the obstacle ω_k,
  - set $\Omega_{k+1}=\Omega_k\backslash\overline{\omega_k}$,
- $k\longleftarrow k+1$.

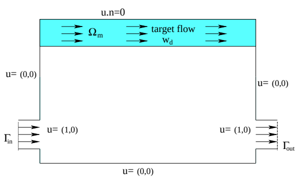
This algorithm can be seen as a descent method where the descent direction is determined by the topological sensitivity g_k and the step length is given by the volume variation $meas(\Omega_k\backslash\Omega_{k+1})$. The natural optimality condition $g_k(x)\geq0,\forall x\in\mathcal{O}_k$ is used as stopping criteria [4].

Approximation of a wanted flow. The aim is to determine the optimal shape $\mathcal{O}^*\subset\Omega$ of the fluid flow domain such that the velocity $u_{\mathcal{O}^*}$, solution to the Stokes equations in $\mathcal{O}^*$, approximate a wanted flow w_d defined in a fixed domain $\Omega_m\subset\Omega$. The optimal shape $\mathcal{O}^*$ can be characterized as the solution to the following topological optimization problem

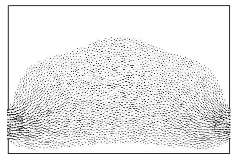
$$\min_{\mathcal{O}\subset\Omega}\int_{\Omega_m}|u_{\mathcal{O}}-w_d|^2\mathrm{dx},$$

where $u_{\mathcal{O}}$ is the solution to the Stokes equations in $\mathcal{O}\subset\Omega$. This test is treated in two and three dimensional cases. In 2D, the tank $\Omega=[0,1.5]\times$

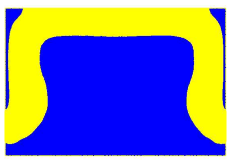
$[0, 1]$, the domain $\Omega_m = [0, 1.5] \times [0.8, 1]$ and the velocity field w_d is defined by: $w_d = (1, 0)$ in Ω_m and $w_d = (0, 0)$ elsewhere. The numerical results are described in Figure 1. A 3D extension of this case is presented in Figure 2.



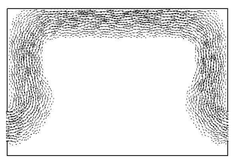
(a) The initial geometry Ω



(b) The velocity field in the initial domain

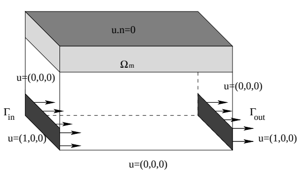


(c) The optimal domain is obtained in only 3 iterations

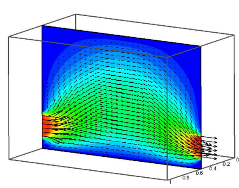


(d) The velocity field in the obtained domain

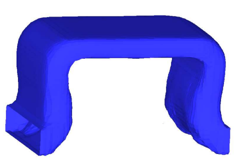
Fig. 1. Approximation of a wanted flow: 2D case



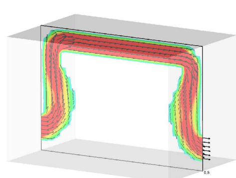
(a) The initial geometry



(b) The velocity field in the initial domain



(c) The optimal domain is obtained in only 4 iterations



(d) The velocity field in the obtained domain

Fig. 2. Approximation of a wanted flow: 3D case

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