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Subordination formulae for space-time fractional diffusion processes via Mellin convolution

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***Abstract*—Fundamental solutions of space-time fractional diffusion equations can be interpret as probability density functions. This fact creates a strong link with stochastic processes. Recasting probability density functions in terms of subordination laws has emerged to be important to built up stochastic processes. In particular, for diffusion processes, subordination can be understood as a diffusive process in space, which is called parent process, that depends on a parameter which is also random and depends on time, which is called directing process. Stochastic processes related to fractional diffusion are self-similar processes. The integral representation of the resulting probability density function for self-similar stochastic processes can be related to the convolution integral within the Mellin transform theory. Here, subordination formulae for space-time fractional diffusion are provided. In particular, a noteworthy new formula is derived in the diffusive symmetric case that is spatially driven by the Gaussian density. Future developments of the research on the basis of this new subordination law are discussed.**

I. Introduction

Fractional diffusion processes are processes governed by diffusion-type equations including fractional differential operators [1], [2]. In particular, Fractional Calculus has emerged to be a useful mathematical tool for modelling non-local effects and then, for what concerns diffusion, to model those phenomena for which the classical local flux-gradient relationship does not hold and a non-local relationship is required. Since these difference from classical/*normal* diffusion, such processes are referred to as *anomalous* diffusion processes.

The resulting diffusion process differs from classical diffusion because the probability density function (PDF) of particle distribution is not Gaussian and because the variance of particle spreading does not grow linearly in time. Anomalous diffusion has been experimentally established in nature in several phenomena, see e.g. [3], [4], [5], [6], [7].

Mellin transform theory has an important role in the study of space-time fractional diffusion equations especially to derive solution in terms of the Mellin–Barnes integral representation [8], [9], [10]. Furthermore, Mellin transform is also intrinsically related to probability theory. In fact, the PDF of the product of two independent random variables is determined by the Mellin convolution of the two corresponding densities [9], [11]. The resulting integral formula can be seen also as a subordination law. The relationship between them is reported.

Hence certain integral formulae for fundamental solutions of space-time fractional diffusion equation can be read as subordination laws. Here manipulation of such formulae is performed with the final aim to obtain a new formula, for the spatial symmetric case, with the valuable property to be based on the Gaussian density. And backing to Mellin convolution, suggestions for future research development to generate stochastic processes by the product of two independent random variables are addressed.

The paper is organized as follows. In Section II the space-time fractional diffusion equation is reviewed in detail discussing the probability density interpretation of the Green functions and showing solutions and special cases. In Section III the essential notions and notations concerning Mellin transform are reported and the relationship with subordination laws highlighted. In Section IV subordination laws for fundamental solution of space-time fractional diffusion are given and a new formula for symmetric diffusion is derived whose parent process is the Gaussian density. Finally, Section V contains summary, conclusions and future developments.

II. The space-time fractional diffusion equation

Space-time fractional diffusion equation is obtained from the ordinary diffusion equation by replacing the first order time derivative with the *Caputo* time-fractional derivative of order β, i.e. ${}_tD_*^\beta$, and the second order space derivative with the *Riesz–Feller* space-fractional derivative of order α and asymmetry parameter θ, i.e. ${}_xD_\theta^\alpha$, [8]

$$ {}_tD_*^\beta \, u(x;t) = {}_xD_\theta^\alpha \, u(x;t) \, , \quad x \in R \, , \quad t \in R_0^+ \, . \qquad (1)$$

The real parameters α, θ and β are restricted as follows

$$\begin{cases} 0 < \alpha \le 2\,, \\ |\theta| \le \min\{\alpha, 2-\alpha\}\,, \\ 0 < \beta \le 1 \quad \text{or} \quad 1 < \beta \le \alpha \le 2\,. \end{cases} \tag{2}$$

The *Caputo* time-fractional derivative ${}_tD_*^{\beta}$ is defined by its Laplace transform as

$$\int_0^{+\infty} \mathrm{e}^{-st} \left\{ {}_tD_*^{\beta}\, u(x;t) \right\} dt = s^{\beta}\, \widetilde{u}(x;s) - \sum_{n=0}^{m-1} s^{\beta-1-n}\, u^{(n)}(x;0^+)\,, \tag{3}$$

with $m-1 < \beta \le m$ and $m \in I\!N$.

The *Riesz–Feller* space-fractional derivative ${}_xD_\theta^{\theta}$ is defined by its Fourier transform according to

$$\int_{-\infty}^{+\infty} \mathrm{e}^{+i\kappa x} \left\{ {}_xD_\theta^{\alpha}\, u(x;t) \right\} dx = -|\kappa|^{\alpha}\, \mathrm{e}^{i(\operatorname{sign}\kappa)\theta\pi/2}\, \widehat{u}(\kappa;t)\,, \tag{4}$$

with α and θ as stated in (2).

In literature the time-fractional derivative is sometimes considered in the Riemann–Liouville sense, here denoted by ${}_tD^{\beta}$. Its relationship with the time-fractional derivative in the Caputo sense is [12]

$${}_tD_*^{\beta}\, u(x;t) = {}_tD^{\beta}\, u(x;t) - \frac{t^{-\beta}}{\Gamma(1-\beta)}\, u(x;0)\,, \tag{5}$$

and equation (1) becomes

$${}_tD^{\beta}\, u(x;t) = {}_xD_\theta^{\alpha}\, u(x;t) + \frac{t^{-\beta}}{\Gamma(1-\beta)}\, u(x;0)\,, \tag{6}$$

with $x \in R$ and $t \in R_0^+$. Equation (1) is stated also as

$$\frac{\partial u}{\partial t} = {}_tD^{1-\beta} \left[{}_xD_\theta^{\alpha}\, u(x;t) \right]\,. \tag{7}$$

However, it is possible to show that the fundamental solutions of (1), (6) and (7) are equal [12].

Solution of (1) can be determined in terms of the fundamental solution, or Green function, $K_{\alpha,\beta}^{\theta}(x;t)$ as follows

$$u(x;t) = \int_{-\infty}^{+\infty} K_{\alpha,\beta}^{\theta}(x-\xi;t)\, u(\xi;0)\, d\xi\,, \tag{8}$$

with the initial and boundary conditions $\{u(x;0) = \delta(x)\,, u_t(x;0) = 0\}$ when $0 < \beta \le 1$ and when $1 < \beta \le 2$ a second initial condition corresponding to $u_t(x;0) = \left.\dfrac{\partial u}{\partial t}\right|_{t=0}$ is needed such that two Green functions follow according to the conditions $\{u(x;0) = \delta(x)\,, u_t(x;0) = 0\}$ and $\{u(x;0) = 0\,, u_t(x;0) = \delta(x)\}$, respectively.

By taking into account the Laplace transform for the *Caputo* time fractional derivative (3) and the Fourier transform for the *Riesz–Feller* space fractional derivative (4), the composite Fourier–Laplace transform of the *first* Green function results to be

$$\widehat{\widetilde{K_{\alpha,\beta}^{\theta}}}(\kappa;s) = \frac{s^{\beta-1}}{s^{\beta} + |\kappa|^{\alpha}\, \mathrm{e}^{i(\operatorname{sign}\kappa)\theta\pi/2}}\,, \tag{9}$$

and of the *second* Green function

$$\widehat{\widetilde{K_{\alpha,\beta}^{\theta}}}(\kappa;s) = \frac{s^{\beta-2}}{s^{\beta} + |\kappa|^{\alpha}\, \mathrm{e}^{i(\operatorname{sign}\kappa)\theta\pi/2}}\,. \tag{10}$$

From (9), for the first Green function it holds

$$\widehat{\widetilde{K_{\alpha,\beta}^{\theta}}}(0;s) = 1/s \quad \text{and then} \quad \widehat{K_{\alpha,\beta}^{\theta}}(0;t) = 1\,, \tag{11}$$

so that the *normalization property* follows

$$\int_{-\infty}^{+\infty} K_{\alpha,\beta}^{\theta}(x;t)\, dx = 1\,. \tag{12}$$

Hence it can be interpret as a PDF for particle distribution in x and evolving in time t. Differently, from (10), for the second Green function it holds

$$\widehat{\widetilde{K_{\alpha,\beta}^{\theta}}}(0;s) = 1/s^2 \quad \text{and then} \quad \widehat{K_{\alpha,\beta}^{\theta}}(0;t) = t\,, \tag{13}$$

so that it follows

$$\int_{-\infty}^{+\infty} K_{\alpha,\beta}^{\theta}(x;t)\, dx = t\,, \tag{14}$$

and the *normalization property* is not met. The second Green function (10) emerges to be a primitive (with respect to the variable t) of the first Green function (9), so that it cannot be interpreted as a PDF of x evolving in t because it is no longer normalized [13]. Finally, solely the first Green function can be considered for diffusion problems.

In general, fundamental solution $K_{\alpha,\beta}^{\theta}(x;t)$ algebraically decreases as $|x|^{-(\alpha+1)}$, thus it belongs to the domain of attraction of the Lévy stable densities of index α. Moreover, $K_{\alpha,\beta}^{\theta}(x;t)$ self-similarly scales as

$$K_{\alpha,\beta}^{\theta}(x;t) = t^{-\beta/\alpha}\, K_{\alpha,\beta}^{\theta}\left(\frac{x}{t^{\beta/\alpha}} \right)\,, \tag{15}$$

and it meets the following *symmetry relation*

$$K_{\alpha,\beta}^{\theta}(-x;t) = K_{\alpha,\beta}^{-\theta}(x;t)\,, \tag{16}$$

which allows the restriction of the analysis to $x \in R_0^+$. In this x-range, i.e. $x \in R_0^+$, the analytical solution of (1) can be expressed by the following Mellin–Barnes integral representation [14], [8],

$$K_{\alpha,\beta}^{\theta}(x;t) = \frac{1}{\alpha x} \times \frac{1}{2\pi i} \int_{\omega-i\infty}^{\omega+i\infty} \frac{\Gamma\left(\frac{s}{\alpha}\right) \Gamma\left(1-\frac{s}{\alpha}\right) \Gamma(1-s)}{\Gamma\left(1-\frac{\beta}{\alpha}s\right) \Gamma(\rho\, s) \Gamma(1-\rho\, s)} \left(\frac{x}{t^{\beta/\alpha}} \right)^{s} ds\,, \tag{17}$$

where $\rho = \dfrac{\alpha-\theta}{2\,\alpha}$ and ω is a suitable real constant. Solution (17) can be also expressed in terms of H-Fox function [14], [15].

The special cases of space-time fractional diffusion equation (1) are the following.

The *space-fractional diffusion* equation, i.e. $0 < \alpha < 2$ and $\beta = 1$, so that when $x \in R_0^+$

$$K_{\alpha,1}^{\theta}(x;t) = L_{\alpha}^{\theta}(x;t) = t^{-1/\alpha} L_{\alpha}^{\theta}\left(\frac{x}{t^{1/\alpha}}\right), \qquad (18)$$

where $L_{\alpha}^{\theta}(x)$ is the class of strictly stable densities with algebraic tail decaying as $|x|^{-(\alpha+1)}$ and infinite variance. Moreover, a stable PDF with $0 < \alpha < 1$ and extremal value of the asymmetry parameter θ are one-sided with support R_0^+ if $\theta = -\alpha$ and R_0^- if $\theta = +\alpha$.

The *time-fractional diffusion* equation, i.e. $\alpha = 2$ and $0 < \beta < 2$, so that when $x \in R_0^+$

$$K_{2,\beta}^{0}(x;t) = \frac{1}{2} M_{\beta/2}(x;t) = \frac{1}{2} t^{-\beta/2} M_{\beta/2}\left(\frac{x}{t^{\beta/2}}\right), \quad (19)$$

where $M_{\nu}(x)$, $0 < \nu < 1$, is the M-Wright/Mainardi density [16], [17], [18], [19], [20], [21], [18], which has stretched exponential tails and finite variance proportional to t^{β}. Since $\alpha = 2$, according to (2), it holds $\theta = 0$, then the PDF is symmetric and the extension to $x \in R$ is obtained by replacing x with $|x|$ in (19).

The *neutral fractional diffusion* equation, i.e. $0 < \alpha = \beta < 2$, whose solution can be expressed in explicit form by non-negative simple elementary functions [22], [8], so that when $x \in R_0^+$

$$\begin{aligned} K_{\alpha,\alpha}^{\theta}(x;t) &= N_{\alpha}^{\theta}(x;t) \\ &= \frac{t^{-1}}{\pi} \frac{(x/t)^{\alpha-1} \sin[\frac{\pi}{2}(\alpha-\theta)]}{1 + 2(x/t)^{\alpha} \cos[\frac{\pi}{2}(\alpha-\theta)] + (x/t)^{2\alpha}}. \quad (20) \end{aligned}$$

Recently Luchko [23] has considered and analyzed the case $1 < \alpha < 2$ and $\theta = 0$ of (20). Moreover, PDF (20) with $0 < \alpha < 1$ is emerged in the study of finite Larmor radius effects on non-diffusive tracer transport in a zonal flow [24] as well as numerically evidenced in non-diffusive chaotic transport by Rossby waves in zonal flow [25].

The *classical diffusion* equation, i.e. $\alpha = 2$ and $\beta = 1$, whose Gaussian solution is recovered as limiting case from both the space-fractional ($\alpha = 2$) and the time-fractional ($\beta = 1$) diffusion equation, so that when $x \in R_0^+$

$$\begin{aligned} K_{2,1}^{0}(x;t) = L_2^0(x;t) &= \frac{1}{2} M_{1/2}(x;t) \\ &= \mathcal{G}(x;t) = \frac{\mathrm{e}^{-x^2/(4t)}}{\sqrt{4\pi t}}. \quad (21) \end{aligned}$$

The last special case is the limit case of the *D'Alembert wave equation*, i.e. $\alpha = \beta = 2$, such that when $x \in R_0^+$ it holds

$$K_{2,2}^{0}(x;t) = \frac{1}{2} M_1(x;t) = \frac{1}{2} \delta(x - t). \qquad (22)$$

III. Mellin convolution and subordination law in stochastic processes

A. The Mellin transform

Main definitions and formulae of Mellin transform are here reminded for completeness with what follows. The interest reader can find exhaustive presentation of the Mellin transform theory, for example, in the book by Marichev [26] where connections with Fourier and Laplace transforms are also reported. However, the theory of Mellin transform independently of Laplace or Fourier transforms was introduced by Butzer and Jansche [27], [28].

Let $f(x) \in L_{loc}(I\!R^+)$, then the Mellin transform $f^*(s)$, $s \in \mathbb{C}$, of a sufficiently well-behaved function $f(x)$, $x \in I\!R^+$, is defined as

$$f^*(s) = \int_0^{+\infty} f(x)\, x^{s-1}\, dx, \quad s \in \mathbb{C}. \qquad (23)$$

Formula (23) defines the Mellin transform in a vertical strip in the s-plane whose boundaries are determined by the analytic structure of $f(x)$ as $x \to 0^+$ and $x \to +\infty$. When

$$f(x) = \begin{cases} O\left(x^{-\omega_1-\epsilon}\right) & \text{as} \quad x \to 0^+, \\ O\left(x^{-\omega_2-\epsilon}\right) & \text{as} \quad x \to +\infty, \end{cases} \qquad (24)$$

then, for every (small) $\epsilon > 0$ and $\omega_1 < \omega_2$, integral (23) converges absolutely and defines an analytic function in the strip $\omega_1 < \mathrm{Re}\{s\} < \omega_2$, i.e. the *strip of analyticity* of $f^*(s)$.

The inversion formula follows directly from the inversion formula for the bilateral Laplace transform, i.e.

$$f(x) = \frac{1}{2\pi i} \int_{\omega-i\infty}^{\omega+i\infty} f^*(s)\, x^{-s}\, ds, \quad \omega_1 < \omega < \omega_2, \quad (25)$$

when $f(x)$ is continuous.

Denoting by $\overset{\mathcal{M}}{\longleftrightarrow}$ the juxtaposition of a function $f(x)$, $x \in I\!R^+$, with its Mellin transform $f^*(s)$, $s \in \mathbb{C}$, some important properties of Mellin transform are

$$x^a f(x) \overset{\mathcal{M}}{\longleftrightarrow} f^*(s+a), \quad a \in \mathbb{C}, \qquad (26)$$

$$f(x^b) \overset{\mathcal{M}}{\longleftrightarrow} \frac{1}{|b|} f^*(s/b), \quad b \in \mathbb{C}, \quad b \neq 0, \qquad (27)$$

$$f(cx) \overset{\mathcal{M}}{\longleftrightarrow} c^{-s} f^*(s), \quad c \in I\!R^+, \qquad (28)$$

from which it follows

$$x^a f(cx^b) \overset{\mathcal{M}}{\longleftrightarrow} \frac{1}{|b|} c^{-(s+a)/b} f^*\left(\frac{s+a}{b}\right). \qquad (29)$$

Differently from Fourier and Laplace transforms, *Mellin convolution* formula does not concern variable shifting but scaling and it emerges to be

$$h(x) = \int_0^{\infty} f\left(\frac{x}{\xi}\right) g(\xi) \frac{d\xi}{\xi} \overset{\mathcal{M}}{\longleftrightarrow} f^*(s)\, g^*(s) = h^*(s). \quad (30)$$

In general, formula (30) can be rewritten for $\gamma > 0$ as

$$\begin{aligned} h(x) &= \int_0^{\infty} f\left(\frac{x}{\xi^{\gamma}}\right) g(\xi) \frac{d\xi}{\xi^{\gamma}} \\ &\overset{\mathcal{M}}{\longleftrightarrow} f^*(s)\, g^*[\gamma(s-1)+1] = h^*(s). \quad (31) \end{aligned}$$

Formula (31) embodies the operative tool of the following analysis.

B. Subordination law

A stochastic process $X(t)$ is called *subordinated process* if it is obtained by a stochastic process $Y(\tau)$, $\tau \in I\!R_0^+$, by the randomization of the parameter τ according to a process $T(t)$ with non-negative independent increments [29]. The resulting process $X(t) = Y(T(t))$ is said to be subordinated to $Y(t)$, that is called the *parent process*, and to be directed by $T(t)$, that is called the *directing process*.

The subordinated process $X(t) = Y(T(t))$ emerges to be governed by a spatial PDF of x evolving in time t, i.e. $\psi(x;t)$, determined by the subordination law

$$\psi(x;t) = \int_0^\infty q(x;\tau)\,\varphi(\tau;t)\,d\tau\,, \qquad (32)$$

where $q(x;\tau)$ is the spatial PDF of x depending on the parameter τ and corresponding to the process $Y(\tau)$, $\varphi(\tau;t)$ is the PDF underlying the process $T(\tau)$ with non-negative independent increments and depending on the parameter t.

Assuming self-similarity for the parent process $Y(\tau)$ then it holds

$$q(x;\tau) = \tau^{-\gamma}\,q\Big(\frac{x}{\tau^\gamma}\Big)\,, \quad \gamma > 0\,, \qquad (33)$$

and formula (32) reads

$$\psi(x;t) = \int_0^\infty q\Big(\frac{x}{\tau^\gamma}\Big)\,\varphi(\tau;t)\,\frac{d\tau}{\tau^\gamma}\,, \quad \gamma > 0\,. \qquad (34)$$

Moreover, let Z_1 and Z_2 be two real *independent* random variables with PDFs $p_1(z_1)$, $z_1 \in I\!R$, and $p_2(z_2)$, $z_2 \in I\!R_0^+$, respectively. From theory of probability it follows that the joint probability $p(z_1,z_2)$ is

$$p(z_1,z_2) = p_1(z_1)\,p_2(z_2)\,. \qquad (35)$$

Introducing the change of variables

$$\begin{cases} z_1 = z/\lambda^\gamma\,, \\ z_2 = \lambda\,, \end{cases} \qquad (36)$$

whose Jacobian equals $1/\lambda^\gamma$, it follows that

$$p(z,\lambda)\,dz\,d\lambda = p_1\left(\frac{z_1}{\lambda^\gamma}\right)\,p_2(\lambda)\frac{d\lambda}{\lambda^\gamma}\,dz\,, \qquad (37)$$

and integrating in $d\lambda$ the PDF of $Z = Z_1 Z_2^\gamma$ finally turns out to be

$$p(z) = \int_{-\infty}^{+\infty} p_1\left(\frac{z}{\lambda^\gamma}\right)\,p_2(\lambda)\,\frac{d\lambda}{\lambda^\gamma}\,. \qquad (38)$$

When $\gamma = 1$ formula (38) corresponds to Mellin convolution integral (30) and in general, when $\gamma \neq 1$, it is equal to (31) proving the fact that Mellin convolution is related to the PDF resulting from the product of two independent random variables.

Clearly, by making the change of variables $z = x\,t^{-\gamma\Omega}$ and $\lambda = \tau\,t^{-\Omega}$ and by setting $\tau^{-\gamma}p_1(x/\tau^\gamma) \equiv \tau^{-\gamma}q(x/\tau^\gamma)$ and $t^{-\Omega}p_2(\tau/t^\Omega) \equiv \varphi(\tau;t)$, from (38) formula (34) is recovered and it holds $t^{-\gamma\Omega}p(x/t^{\gamma\Omega}) \equiv \psi(x;t)$.

Hence, the stochastic process $X(t) = Y(T(t))$ follows the same one-point one-time PDF of the process $X = X_1 X_2^\gamma$. Then for a given diffusion equation, a stochastic process corresponding to the Green function can be generated by the product of two independent random variables distributed according to the PDFs involved in the subordination law.

This approach to provide stochastic processes has been recently discussed by the author and collaborators [30]. In particular this method has been introduced to develop self-similar stochastic processes with stationary increments following a method proposed by Mura [31] to derive the so-called generalized grey Brownian motion [32], [33]. Actually, the generalized grey Brownian motion has been shown to be related to the Green function of the Erdélyi–Kober fractional diffusion [34], [35], [36].

IV. Subordination laws for the space-time fractional diffusion

A first valuable subordination-type formula for $K^\theta_{\alpha,\beta}(x;t)$ was derived by Uchaikin & Zolotarev [37], [38], i.e.

$$K^\theta_{\alpha,\beta}(x;t) = \int_0^\infty L^\theta_\alpha(x;(t/y)^\beta)\,L^{-\beta}_\beta(y)\,dy\,, \qquad (39)$$

and, by putting $t/y = \xi^{1/\beta}$, it becomes [8]

$$K^\theta_{\alpha,\beta}(x;t) = \int_0^\infty L^\theta_\alpha(x;\xi)\,L^{-\beta}_\beta(t;\xi)\,\frac{t}{\beta\,\xi}\,d\xi\,. \qquad (40)$$

Further important subordination formulae were derived in literature. A practical method is the following. Noting the close relationship between Mellin–Barnes integral representation of fundamental solutions (17) and Mellin inversion formula (25), by splitting Mellin transform of Green functions in two known Mellin transforms then subordination laws can be derived by using Mellin convolution formula (30, 31) [8], [9], [11]. The same method was used also to obtain a Gaussianization of Lévy noise in signal filtering [39].

In particular, it is reported that when $0 < \beta \le 1$ it holds [8, equation (6.16)]

$$K^\theta_{\alpha,\beta}(z) = \alpha\int_0^\infty \xi^{\alpha-1}\,M_\beta\left(\xi^\alpha\right)\,L^\theta_\alpha(z/\xi)\,\frac{d\xi}{\xi}\,, \qquad (41)$$

and when $0 < \beta/\alpha \le 1$

$$K^\theta_{\alpha,\beta}(z) = \int_0^\infty M_{\beta/\alpha}(\xi)\,N^\theta_\alpha(z/\xi)\,\frac{d\xi}{\xi}\,. \qquad (42)$$

Applying the changes of variable $\xi = \tau^{1/\alpha}/t^{\beta/\alpha}$ and $\xi = \tau/t^{\beta/\alpha}$ in (41) and (42), respectively, and replacing z with $x/t^{\beta/\alpha}$, when $0 < \beta \le 1$ it follows [8], [9],

$$t^{-\beta/\alpha}K^\theta_{\alpha,\beta}\left(\frac{x}{t^{\beta/\alpha}}\right) = \int_0^\infty \tau^{-1/\alpha}L^\theta_\alpha\left(\frac{x}{\tau^{1/\alpha}}\right)t^{-\beta}M_\beta\left(\frac{\tau}{\beta}\right)d\tau\,, \qquad (43)$$

and when $0 < \beta/\alpha \le 1$

$$t^{-\beta/\alpha}K^\theta_{\alpha,\beta}\left(\frac{x}{t^{\beta/\alpha}}\right) = \int_0^\infty \tau^{-1}N^\theta_\alpha\left(\frac{x}{\tau}\right)t^{-\beta/\alpha}M_{\beta/\alpha}\left(\frac{\tau}{t^{\beta/\alpha}}\right)d\tau\,, \qquad (44)$$

or analogously when $0 < \beta \le 1$ [8], [9],

$$K^\theta_{\alpha,\beta}(x;t) = \int_0^\infty L^\theta_\alpha(x;\tau)\, M_\beta(\tau;t)\, d\tau\,, \tag{45}$$

and when $0 < \beta/\alpha \le 1$

$$K^\theta_{\alpha,\beta}(x;t) = \int_0^\infty N^\theta_\alpha(x;\tau)\, M_{\beta/\alpha}(\tau;t)\, d\tau\,. \tag{46}$$

The symmetric case, i.e. $\theta = 0$, of formula (45) was previously derived by Saichev & Zavlasky [22]. Moreover, combining (40) and (45) for $0 < \beta \le 1$, with $\tau\,, t \in I\!R^+_0$, it follows the identity [8]

$$L^{-\beta}_\beta(t;\tau)\, \frac{t}{\beta\,\tau} = M_\beta(\tau;t)\,, \tag{47}$$

and by using self-similarity properties

$$L^{-\beta}_\beta\left(\frac{t}{\tau^{1/\beta}}\right)\, \frac{t}{\beta\,\tau^{1/\beta+1}} = \frac{1}{t^\beta}\, M_\beta\left(\frac{\tau}{t^\beta}\right)\,, \tag{48}$$

with $0 < \beta \le 1$ and $\tau\,, t \in I\!R^+_0$.

Since L^θ_α, M_ν and N^θ_α are special cases of $K^\theta_{\alpha,\beta}$, see Section II, subordination formulae (45) and (46) can be restated also in terms of $K^\theta_{\alpha,\beta}(x;t)$ only, after the opportune choice of parameters [8], [9]. In fact when $0 < \beta \le 1$

$$K^\theta_{\alpha,\beta}(x;t) = 2\int_0^\infty K^\theta_{\alpha,1}(x;\tau)\, K^0_{2,2\beta}(\tau;t)\, d\tau\,, \tag{49}$$

and when $0 < \beta/\alpha \le 1$

$$K^\theta_{\alpha,\beta}(x;t) = 2\int_0^\infty K^\theta_{\alpha,\alpha}(x;\tau)\, K^0_{2,2\beta/\alpha}(\tau;t)\, d\tau\,. \tag{50}$$

Formula (45), or the analog ones, shows that the solution of the space-time fractional diffusion equation (1) can be expressed in terms of the solution of the space-fractional diffusion equation of order α, i.e. $K^\theta_{\alpha,1}(x;t) = L^\theta_\alpha(x;t)$, and of the solution of the time-fractional diffusion equation of order 2β, i.e. $K^0_{2,2\beta}(\tau;t) = M_\beta(\tau;t)/2$, $\tau \in R^+_0$. Moreover, formulae (45) and (46), or the analog ones, by involving non-negative functions allow the PDF interpretation of $K^\theta_{\alpha,\beta}(x;t)$. Furthermore, it is worth-noting to remark that formula (46), or the analog ones, is fundamental to extend such probability interpretation to the range $1 < \beta \le \alpha \le 2$.

By using (45) a new subordination law for the space-time fractional diffusion can be derived. In fact, it is well known that the following subordination formula for Lévy stable density holds [29], [9], [11],

$$L^\theta_\alpha(x;t) = \int_0^\infty L^\gamma_\eta(x;\xi)\, L^{-\nu}_\nu(\xi;t)\, d\xi\,, \tag{51}$$

where $\alpha = \eta\nu$, $\theta = \gamma\nu$ and

$$0 < \alpha \le 2\,, \quad |\theta| \le \min\{\alpha, 2-\alpha\}\,, \tag{52}$$

$$0 < \eta \le 2\,, \quad |\gamma| \le \min\{\eta, 2-\eta\}\,, \quad 0 < \nu \le 1\,. \tag{53}$$

Hence, inserting (51) into (45) gives

$$\begin{aligned} K^\theta_{\alpha,\beta}(x;t) &= \int_0^\infty \left\{ \int_0^\infty L^\gamma_\eta(x;\xi) L^{-\nu}_\nu(\xi;\tau) d\xi \right\} M_\beta(\tau;t) d\tau\,, \\ &= \int_0^\infty L^\gamma_\eta(x,\xi) \left\{ \int_0^\infty L^{-\nu}_\nu(\xi;\tau) M_\beta(\tau;t)\, d\tau \right\} d\xi\,, \end{aligned} \tag{54}$$

where the exchange of integration is allowed by the fact that the involved functions are normalized PDFs. Finally, using again (45) to compute the integral into braces in (54), when $0 < \beta \le 1$, the following *new* subordination law is obtained

$$K^\theta_{\alpha,\beta}(x;t) = \int_0^\infty L^\gamma_\eta(x;\xi)\, K^{-\nu}_{\nu,\beta}(\xi,t)\, d\xi\,, \tag{55}$$

with $\alpha = \eta\nu$, $\theta = \gamma\nu$ and the same restrictions stated in (52) and (53) for the values of parameters.

In the particular case $\eta = 2$ and $\gamma = 0$, so that $\nu = \alpha/2$ and $\theta = 0$, a Gaussian subordination follows. In fact $L^0_2(x;t) = \mathcal{G}(x;t) = \dfrac{\mathrm{e}^{-x^2/(4\,t)}}{\sqrt{4\,\pi\,t}}$ so that formula (55) becomes

$$K^0_{\alpha,\beta}(x;t) = \int_0^\infty \mathcal{G}(x;\xi)\, K^{-\alpha/2}_{\alpha/2,\beta}(\xi;t)\, d\xi\,, \tag{56}$$

with $0 < \alpha \le 2$ and $0 < \beta \le 1$.

V. Conclusion

In the present paper fundamental solutions of space-time fractional diffusion equations have been analysed. In particular, by using Mellin–Barnes integral representation and Mellin convolution, integral formulae can be derived that may be understood as subordination laws. It is well known that Mellin convolution gives the integral representation of the PDF resulting from the product of two independent random variables. Then subordination laws suggest how to built up stochastic processes by using the product of two independent variables that follows a desired one-point one-time PDF.

Manipulation of literature formulae has been performed with the aim to obtain a new subordination-type formula for space-time fractional diffusion. The derived new formula, when reduced to the case of spatial symmetric diffusion, has emerged to be based on the Gaussian density. That is a valuable property.

In fact, the derived formula proves that stochastic processes, whose one-point one-time PDF is solution of the symmetric space-time fractional diffusion equation, can be generated by the product of a Gaussian distributed motion and an independent positive random variable with specified PDF. The leading role of the Gaussian motion is remarkable even because it is a very well studied process and largely suitable for the simulation of trajectories. In particular because self-similar with stationary increments and characterized by solely the first and the second moments.

To conclude, the derived subordination formula is the basis for future development of a self-similar stochastic process with stationary increments to model space-time fractional diffusion in the spatial symmetric case.

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