Operational methods for Hermite polynomials

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Abstract— We exploit methods of operational nature to derive a set of new identities involving families of polynomials associated with operators providing different realizations of the Weyl group. The identities, we will deal with, extend the Nielsen formulae, valid for ordinary Hermite to families of Hermite-like polynomials. It will also be shown that the underlying formalism yields the possibility of obtaining further identities relevant to multi-variable and multi-index polynomials.

Keywords— Orthogonal Polynomials, Hermite, Weyl group, monomiality principle, generating functions.

I. Introduction

The use of the *monomiality principle* [1], a by product of the Lie group treatment of special functions [2,3], has offered a powerful tool for studying the properties of families of special functions and polynomials. Within the context of such a treatment, a polynomial $p_n(x)$ is said *quasi-monomial* (*q.m.*), if two operators exist and act on the polynomial as a derivative and multiplicative operators respectively, namely:

$$\begin{aligned} &\hat{M}\, p_n(x) = p_{n+1}(x) \\ &\hat{P}\, p_n(x) = n p_{n-1}(x). \end{aligned} \tag{1}$$

In the case of two-variable Kampé de Fériet polynomials [1,4,5], we have:

$$H_n(x,y) = n! \sum_{r=0}^{[n/2]} \frac{y^r x^{n-2r}}{r!(n-2r)!} = (-i\sqrt{y})^n H_n\left(\frac{ix}{2\sqrt{y}}\right), \tag{2}$$

where the associated multiplication and derivative operators, are identified as:

$$\begin{aligned} &\hat{M} = x + 2y\frac{\partial}{\partial x} \\ &\hat{P} = \frac{\partial}{\partial x}. \end{aligned} \tag{3}$$

According to what has been discussed in reference [1], if $p_0(x) = 1$, then $p_n(x)$ can be explicitly constructed as:

$$\hat{M}^n (1) = p_n(x), \tag{4}$$

thus, in the case of Hermite, we obtain:

$$\begin{aligned} &\left(x + 2y\frac{\partial}{\partial x}\right)^n (1) = \sum_{r=0}^{n} \binom{n}{r} (2y)^r H_{n-r}(x,y) \left(\frac{\partial}{\partial x}\right)^r (1) = \\ &= H_n(x,y). \end{aligned} \tag{5}$$

The above identity is essentially the Burchnall operational formula, whose proof can be found in the papers [6,7]; in the next section, where the problem is treated in a wider context, we will see a generalization of this identity.

By using the above relations, we can immediately state the following identity:

$$H_{2n}(x,y) = \left(x + 2y\frac{\partial}{\partial x}\right)^n \left(x + 2y\frac{\partial}{\partial x}\right)^n (1). \tag{6}$$

It is easy, in fact to note that the r.h.s. of the equation (6) could be written as:

$$\sum_{r=0}^{n} \binom{n}{r} (2y)^r H_{n-r}(x,y) \left(\frac{\partial}{\partial x}\right)^r H_n(x,y).$$

In the paper [1], we have stated many relevant relations regarding the two-variable Hermite polynomials, in particular it is also possible to obtain the following statement:

$$\frac{\partial^r}{\partial x^r} H_n(x,y) = \frac{n!}{(n-r)!} H_{n-r}(x,y), \tag{7}$$

which is useful to rewrite the relation (6) in the form:

$$H_{2n}(x,y) = (n!)^2 \sum_{r=0}^{n} \frac{(2y)^{n-r} \left[H_r(x,y)\right]^2}{(r!)^2 (n-r)!}. \tag{8}$$

By following an analogous procedure it is possible to derive these relevant relations satisfied by the two-variable Hermite polynomials:

$$H_{n+m}(x,y)=\sum_{r=0}^{[n,m]}\binom{n}{r}\binom{m}{r}r!(2y)^{n-r}H_{n-r}(x,y)H_{m-r}(x,y),$$

$$\left[H_n(x,y)\right]^2=(n!)^2\sum_{r=0}^{n}\frac{(-2y)^{n-r}H_{2r}(x,y)}{(r!)^2(n-r)!}, \tag{9}$$

$$H_n(x,y)H_m(x,y)=n!m!\sum_{r=0}^{[n,m]}\frac{(-2y)^{n-r}H_{n+m-2r}(x,y)}{r!(n-r)!(m-r)!},$$

where we have indicated with $[n,m]$ the minimum of (n,m).
These identities can be viewed as an extension of those derived by Nielsen [2], for the ordinary case.
The paper consists of three sections. In section II we will discuss higher order Kampé de Fériet Hermite polynomials and the associated identities; section III is devoted to final remarks and comments on the possible extension of the method presented to other families recognized as Hermite polynomials.

II. Operational rules and higher order Hermite polynomials

In the paper [4], we have seen the two-variable Hermite polynomials of order $m\in\mathbb{N}$, $m\geq 2$, defined by the series:

$$H_n^{(m)}(x,y)=n!\sum_{r=0}^{[n/m]}\frac{y^r x^{n-mr}}{r!(n-mr)!}. \tag{10}$$

It is immediately easy to observe that these polynomials could be recognized as *quasi-monomial* under the action of the following operators:

$$\hat{M}=x+my\frac{\partial^{m-1}}{\partial x^{m-1}}, \qquad \hat{P}=\frac{\partial}{\partial x}. \tag{11}$$

Moreover, it is possible to introduce a further generalization, by considering the case of *m*-variable Hermite polynomials of order *m*, by setting:

$$H_n^{(m)}(x_1,\ldots,x_m)=n!\sum_{r=0}^{[n/m]}\frac{H_{n-mr}^{(m-1)}(x_1,\ldots,x_m)x_m^r}{r!(n-mr)!}. \tag{12}$$

This family of Hermite polynomials is also *quasi-monomial* with the related operators:

$$\hat{M}=x_1+\sum_{s=2}^{m}sx_s\frac{\partial^{s-1}}{\partial x_1^{s-1}}, \qquad \hat{P}=\frac{\partial}{\partial x}. \tag{13}$$

In the paper [1], presenting the concepts and the related formalism of the *monomiality principle,* we stated the following identity:

$$\hat{M}\hat{P}\,p_n(x)=p_n(x), \tag{14}$$

which implies that the present families of polynomials satisfy the differential equations:

$$\left(my\frac{\partial^m}{\partial x^m}+x\frac{\partial}{\partial x}\right)H_n^{(m)}(x,y)=nH_n^{(m)}(x,y), \tag{15}$$

$$\left(\sum_{s=2}^{m}sx_s\frac{\partial^s}{\partial x_1^s}+x_1\frac{\partial}{\partial x_1}\right)H_n^{(m)}(x_1,\ldots,x_m)=nH_n^{(m)}(x_1,\ldots,x_m). \tag{16}$$

We prove, now, an important extension of the Weyl identity, that is:

$$e^{\xi\left(x+\frac{\partial^n}{\partial x^n}\right)}=e^{x\xi+\frac{\xi^{n+1}}{n+1}}e^{\sum_{r=0}^{(n-1)}\frac{n!\xi^{r+1}}{(n-r)!(r+1)!}\left(\frac{\partial}{\partial x}\right)^{n-r}}, \tag{17}$$

where ξ being a parameter.
If we consider the exponential operator:

$$S\left(\hat{A},\hat{B};\xi\right)=e^{\xi\left(\hat{A}+\hat{B}^n\right)} \tag{18}$$

where ξ is a parameter and $\hat{A}$ and $\hat{B}$ denote operator such that:

$$\left[\hat{A},\hat{B}\right]=\hat{A}\hat{B}-\hat{B}\hat{A}=k,$$

with *k* commuting with both of them.
The decoupling theorem for the exponential operator (18) can be proved as follows. By keeping the derivative of both sides with respect to ξ, we get:

$$\frac{\partial}{\partial\xi}S\left(\hat{A},\hat{B};\xi\right)=\left(\hat{A}+\hat{B}^n\right)S\left(\hat{A},\hat{B};\xi\right), \tag{19}$$

and, after setting:

$$S\left(\hat{A},\hat{B};\xi\right)=e^{\xi\hat{A}}\Sigma,$$

and by using the relation:

$$e^{-\xi\hat{A}}\hat{B}^n e^{\xi\hat{A}}=\left(\hat{B}-\xi k\right)^n,$$

we finally find:

$$\frac{\partial}{\partial \xi}\Sigma = \left(\hat{B} - \xi k\right)^n \Sigma , \tag{20}$$

which can be easily integrated. Thus getting in conclusion:

$$S\left(\hat{A}, \hat{B}; \xi\right) = e^{\xi \hat{A}} e^{\sum_{r=0}^{n} \binom{n}{r} \frac{\hat{B}^{n-r} k^r \xi^{r+1}}{r+1} (-1)^r} . \tag{21}$$

It is immediately to note that identity (17) follows as a particular case with:

$$\hat{A} = x$$
$$\hat{B} = \frac{\partial}{\partial x}$$

The generalization of the Weyl identity, which we have proved above, allows us to derive the following generalized Burchnall identity:

$$\left(x + my\frac{\partial^{m-1}}{\partial x^{m-1}}\right)^n = \sum_{r=0}^{n}\binom{n}{r} H_{n-r}^{(m)}(x,y) H_r^{(m-1)}(x,y)\left(\{G\}_{s=0}^{m-2}\right), \tag{22}$$

where we have indicated with G the expression:

$$G = \frac{m(m-1)!\,y}{(m-1-s)!(s+1)!}\frac{\partial^{m-1-s}}{\partial x^{m-1-s}} .$$

The relation in (22), for $m = 3$, specializes as:

$$\left(x + 3y\frac{\partial^2}{\partial x^2}\right)^2 = \sum_{r=0}^{n}\binom{n}{r} H_{n-r}^{(3)}(x,y) H_r\left(3y\frac{\partial^2}{\partial x^2}, 3y\frac{\partial}{\partial x}\right), \tag{23}$$

An immediate application of these last identities is the derivation of the following Nielsen formula:

$$H_{2n}^{(m)}(x,y) = \sum_{r=0}^{n}\binom{n}{r} H_{n-r}^{(m)}(x,y) F_{n,r}^{(m-1)}(x,y) , \tag{24}$$

where:

$$F_{n,r}^{(m-1)}(x,y) = \\ = H_r^{(m-1)}\left[\left\{\frac{m(m-1)!\,y}{(m-1-r)!(r+1)!}\frac{\partial^{m-1-s}}{\partial x^{m-1-s}}\right\}_{r=0}^{m-2}\right] H_n^{(m)}(x,y). \tag{25}$$

The explicit form of the $F_{n,r}^{(m-1)}(x,y)$ polynomials can be evaluated fairly straightforwardly; in the case $m = 3$, we get indeed:

$$F_{n,r}^{(2)}(x,y) = s!\sum_{r=0}^{[s/2]} \frac{(3y)^{s-r}(2s-3r)!\,H_{n-(2s-3r)}^{(3)}(x,y)}{(s-2r)!\,r!\,[n-(2s-3r)]!} . \tag{26}$$

A further application of the so far developed method is associated with the derivation of generating functions of the type:

$$G_l^{(m)}(x,y;t) = \sum_{n=0}^{+\infty}\frac{t^n}{n!} H_{n+l}^{(m)}(x,y) . \tag{27}$$

Before to proceed, let us remind that [7]:

$$e^{\alpha \frac{\partial^s}{\partial x_1^s}} H_n^{(m)}(x_1,\dots,x_m) = \begin{cases} H_n^{(m)}(x_1,\dots,\alpha + x_s,\dots,x_m), & s \le m \\ H_n^{(m)}(x_1,\dots,x_m,\dots,\alpha), & s > m \end{cases} \tag{28}$$

and then, according with the statement in equation (22), we have:

$$G_l^{(m)}(x,y;t) = \sum_{n=0}^{+\infty}\frac{t^n}{n!}\left(x + my\frac{\partial^{m-1}}{\partial x^{m-1}}\right)^n H_l^{(m)}(x,y) , \tag{29}$$

i.e.

$$G_l^{(m)}(x,y;t) = e^{\left(x + my\frac{\partial^{m-1}}{\partial x^{m-1}}\right)t} H_l^{(m)}(x,y) . \tag{30}$$

Finally, by using the relation (28), we can obtain the relevant operational expression:

$$G_l^{(m)}(x,y;t) = e^{xt + yt^m} H_l^{(m)}\left(x + ymt^{m-1}, \frac{m(m-1)y}{2}t^{m-2}, \dots, y\right). \tag{31}$$

These last results complete the preliminary conclusions obtained in references [5,7]. In the next and last section will be presented further comments on the families of Hermite-like polynomials and will be derived interesting operational rules.

III. Operational rules and multi-index Hermite polynomials

The method described in the previous sections is devoted to the operational rules of polynomials characterized by a single index and, eventually, more than one variable. In this section we will outline the technique to extend the method to multi-index polynomials [8,9,10]. In particular, the structure and some interesting properties of the incomplete 2-dimensional Hermite polynomials, we will consider this family as example to generalize the operational method shown previously.

Let us remind that the incomplete 2-dimensional Hermite polynomials are defined by the series:

$$h_{m,n}(x,y\,|\,\tau)=m!n!\sum_{r=0}^{[m,n]}\frac{\tau^r x^{m-r}y^{n-r}}{r!(m-r)!(n-r)!}, \tag{32}$$

where $\tau\in\mathbb{R}$, $[m,n]=\min(m,n)$ and their generating function has the form:

$$\exp(xu+yv+\tau uv)=\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{u^m}{m!}\frac{v^n}{n!}h_{m,n}(x,y\,|\,\tau). \tag{33}$$

By noting that (see [1,7]), the two-variable Kampé de Fériet Hermite polynomials could be defined also by the following operational expression:

$$H_n(x,y)=e^{y\frac{\partial^2}{\partial x^2}}x^n, \tag{34}$$

it is easy to derive the analogous relation for the polynomials $h_{m,n}(x,y\,|\,\tau)$; we have, indeed:

$$e^{-\tau\frac{\partial^2}{\partial x\partial y}}(x^m y^n)=h_{m,n}(x,y\,|\,\tau). \tag{35}$$

In the first section we have presented the Burchnall identity, see equation (5), and we have stated a generalization for the case of two-variable Hermite polynomials of order *m*, in section II, by equation (22). Before to proceed, it could be useful to exploit the procedure of generalization of this important identity. Let consider the operator:

$$\hat{O}_n=\left(x+my\frac{\partial^{m-1}}{\partial x^{m-1}}\right)^n, \tag{36}$$

by multiplying both sides by:

$\frac{t^n}{n!}$, $t\in\mathbb{R}$ and $n\in\mathbb{N}$

and then by summing up, we get:

$$\sum_{n=0}^{+\infty}\frac{t^n}{n!}\hat{O}_n=e^{t\left(x+my\frac{\partial^{m-1}}{\partial x^{m-1}}\right)}. \tag{37}$$

By using the generalized Weyl identity (eq. (21)), proved in the previous section, we can rearrange the above relation in the form:

$$\sum_{n=0}^{+\infty}\frac{t^n}{n!}\hat{O}_n=e^{xt+yt^m}e^{\sum_{r=0}^{(m-2)}\frac{m!yt^{r+1}}{(m-1-r)!(r+1)!}\frac{\partial^{m-1-r}}{\partial x^{m-1-r}}}, \tag{38}$$

and, since the generating functions of the generalized Hermite polynomials of order *m*, are [1,4]:

$$\begin{aligned}&\sum_{n=0}^{+\infty}\frac{t^n}{n!}H_n^{(m)}(x,y)=e^{xt+yt^m},\\&\sum_{n=0}^{+\infty}\frac{t^n}{n!}H_n^{(m)}(x_1,\dots,x_m)=e^{\sum_{s=1}^{m}x_s t^s},\end{aligned} \tag{39}$$

we can rewritten the r.h.s. of the relation (38) as the product of two series involving the generalized Hermite polynomials of order *m* and we finally obtain:

$$\hat{O}_n=\sum_{s=0}^{n}\binom{n}{s}H_{n-s}^{(m)}(x,y)H_s^{(m-1)}\left(\left\{\frac{m!y}{(m-1-r)!(r+1)!}\frac{\partial^{m-1-r}}{\partial x^{m-1-r}}\right\}_{r=0}^{m-2}\right) \tag{40}$$

which complete prove the generalized Burchnall identity (22).

It is now immediate to derive a further generalization for the incomplete 2-dimensional Hermite polynomials discussed in this section. We can indeed exploit the operational rule stated in the equation (35) to derive the following Burchnall-type identity:

$$\begin{aligned}&\left(x-\tau\frac{\partial}{\partial y}\right)^m\left(y-\tau\frac{\partial}{\partial x}\right)^n=\\&=\sum_{p=0}^{m}\sum_{q=0}^{n}\binom{m}{p}\binom{n}{q}(-\tau)^{p+q}h_{m-p,n-q}(x,y\,|\,\tau)\frac{\partial^p}{\partial y^p}\frac{\partial^q}{\partial x^q}.\end{aligned} \tag{41}$$

In section I, we have stated relevant operational identities for the two-variable Hermite polynomials as presented in the relation (9); it is immediately to note that the polynomials $h_{m,n}(x,y\,|\,\tau)$ satisfied the following identity:

$$h_{2m,2n}(x,y\,|\,\tau)=\left(x-\tau\frac{\partial}{\partial y}\right)^m\left(y-\tau\frac{\partial}{\partial x}\right)^n h_{m,n}(x,y\,|\,\tau), \tag{42}$$

and then, from the formula (41), we can obtain the relevant operational identity :

$$\begin{aligned}&h_{2m,2n}(x,y\,|\,\tau)=\\&(m!n!)^2\sum_{p=0}^{m}\sum_{q=0}^{n}\frac{(-\tau)^{p+q}}{p!q![(m-p)!]^2[(n-q)!]^2}\left[h_{m-p,n-q}(x,y\,|\,\tau)\right]^2.\end{aligned} \tag{43}$$

The aspects and the related considerations presented I this paper could be investigate in a deeper way in a forthcoming investigations. It is important to remark that many of the operational rules presented here could be generalized for a wide range of Hermite-like polynomials. Moreover, the structure of the operational techniques here described is also possible to be extended to other classes of polynomials as the Laguerre and Legendre families. Also about this last point, we will discuss in a future paper.

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