On Asymptotically $\mathcal{I}$-Lacunary statistical Equivalent Sequences of order $\alpha$

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Abstract—This paper presents the following definition which is a natural combination of the definition for asymptotically equivalent of order $\alpha$, where $0<\alpha<1$, $\mathcal{I}$-statistically limit, and $\mathcal{I}$-lacunary statistical convergence. Let $\theta$ be a lacunary sequence; the two nonnegative sequences $x=(x_k)$ and $y=(y_k)$ are said to be asymptotically $\mathcal{I}$-lacunary statistical equivalent of order $\alpha$ to multiple $L$ provided that for every $\varepsilon >0$, and $\delta>0$,

$$\{r \in \mathbb{N} : \frac{1}{h_r} \{ \{ k \in I_r : | \frac{x_k}{y_k} - L \geq \varepsilon \} \geq \delta \} \in \mathcal{I},$$

where $|A|$ denotes the cardinality of $A \subset \mathbb{N}$.

Recently in ([3] and [11]), we used ideals to introduce the concepts of $\mathcal{I}$-statistical convergence and $\mathcal{I}$-lacunary statistical convergence which naturally extend the notions of the above mentioned convergence.

Definition 1. (Marouf, [9]) Two nonnegative sequences $x=(x_k)$ and $y=(y_k)$ are said to be asymptotically equivalent if for arbitrary $\varepsilon >0$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_{r}} | \{ k \in I_{r} : x_{k} - L \geq \varepsilon \} | = 0,$$

where $I_{r}$ is a lacunary sequence and $h_{r}$ is its lacunary ratio.

In this case we write $st \lim x = L$ or $x_{k} \rightarrow L$ (st).

INTRODUCTION

The concept of statistical convergence was introduce by Fast [4] in 1951.

A sequence $(x_k)$ of real numbers is said to be statistically convergent to $L$ if for arbitrary $\varepsilon >0$,

$$\frac{1}{n} \{ \{ k : | x_{k} - L \geq \varepsilon \} \} = 0,$$

where by $k < n$ we mean that $k = 0, 1, 2, \ldots, n$. And the vertical bars indicate the number of elements in the enclosed set. In this case we write $st \lim x = L$ or $x_{k} \rightarrow L$ (st).

The idea of statistical convergence was further extended to $\mathcal{I}$-convergence in [7] using the notion of ideals of $\mathbb{N}$ with many interesting consequences.

Definition 2. (Fridy, [5]) The sequence $x=(x_k)$ has statistical limit $L$, denoted by $st \lim x = L$ provided that for every $\varepsilon >0$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_{r}} \{ \{ k \in I_{r} : x_{k} - L \geq \varepsilon \} \} = 0$$

(\text{denoted by } x \sim y).

Moreover, the following concept is due to Fridy and Orhan[6].

A sequence $(x_k)$ of real numbers is said to be lacunary statistically convergent to $L$ (or $S_\theta$-convergent to $L$) if for any $\varepsilon >0$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_{r}} | \{ k \in I_{r} : x_{k} - L \geq \varepsilon \} | = 0,$$

where $|A|$ denotes the cardinality of $A \subset \mathbb{N}$.
The next definition is natural combination of definitions 1 and 2.

**Definition 3.** (Patterson, [10]) Two nonnegative sequences \( x = (x_k) \) and \( y = (y_k) \) are said to be asymptotically statistical equivalent of multiple \( L \) provided that for every \( \varepsilon > 0 \),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=n}^{\infty} \mathbb{1}_{\{|x_k - L| \geq \varepsilon\}} = 0
\]

(\( \mathbb{1}_{\cdot} \) denotes the number of \( k \leq n \) for which \( |x_k - L| \geq \varepsilon \)).

**Definition 4.** A family \( \mathcal{I} \subset 2^\mathbb{N} \) is said to be an ideal of \( \mathbb{N} \) if the following conditions hold:

(a) \( A, B \in \mathcal{I} \) implies \( A \cup B \in \mathcal{I} \),
(b) \( A \in \mathcal{I}, B \subseteq A \) implies \( B \in \mathcal{I} \).

**Definition 5.** A non-empty family \( F \subset 2^\mathbb{N} \) is said to be an filter of \( \mathbb{N} \) if the following conditions hold:

(a) \( \emptyset \notin F \),
(b) \( A, B \in F \) implies \( A \cap B \in F \),
(c) \( A \in F, A \subseteq B \) implies \( B \in F \).

If \( \mathcal{I} \) is a proper ideal of \( \mathbb{N} \) (i.e., \( \mathbb{N} \notin \mathcal{I} \)), then the family of sets \( F(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \setminus A\} \) is a filter of \( \mathbb{N} \). It is called the filter associated with the ideal.

**Definition 6.** A proper ideal \( \mathcal{I} \) is said to be admissible if \( \{n\} \in \mathcal{I} \) for each \( n \in \mathbb{N} \).

Throughout \( \mathcal{I} \) will stand for a proper admissible ideal of \( \mathbb{N} \).

**Definition 7.** (T") Let \( \mathcal{I} \subset 2^\mathbb{N} \) be a proper admissible ideal of \( \mathbb{N} \). Then the sequence \( (x_k) \) of elements of \( \mathbb{R} \) is said to be \( \mathcal{I} \)-convergent to \( L \in \mathbb{R} \) if for each \( \varepsilon > 0 \) the set \( A(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} \in \mathcal{I} \).

We now introduce our main definitions.

**Definition 8.** A sequence \( x = (x_k) \) is said to be \( \mathcal{I} \)-statistically convergent of order \( \alpha \) to \( L \) or \( S(\mathcal{I})^\alpha \)-convergent to \( L \), where \( 0 < \alpha \leq 1 \), if for each \( \varepsilon > 0 \) and \( \delta > 0 \)

\[
\{n \in \mathbb{N} : \frac{1}{n^\alpha} \sum_{k=n}^{\infty} \mathbb{1}_{\{|x_k - L| \geq \varepsilon\}} \geq \delta\} \in \mathcal{I}.
\]

In this case we write \( x_k \to L(S(\mathcal{I})^\alpha) \). The class of all \( \mathcal{I} \)-statistically convergent sequences of order \( \alpha \) will be denoted by simply \( S(\mathcal{I})^\alpha \).

Also the next definition is natural combination of definitions 1 and 8.

**Definition 9.** The two nonnegative sequences \( x = (x_k) \) and \( y = (y_k) \) are said to be asymptotically \( \mathcal{I} \)-statistical equivalent of order \( \alpha \) to multiple \( L \), where \( 0 < \alpha \leq 1 \), provided that for each \( \varepsilon > 0 \) and \( \delta > 0 \)

\[
\{n \in \mathbb{N} : \frac{1}{n^\alpha} \sum_{k=n}^{\infty} \mathbb{1}_{\{|x_k - L| \geq \varepsilon\}} \geq \delta\} \in \mathcal{I}.
\]

(\( \mathbb{1}_{\cdot} \) denotes by \( x - y \), and simply asymptotically statistical equivalent if \( \mathcal{I} = 1 \).)

**Remark 1.** If \( \mathcal{I} = \mathcal{I}_{\text{f}} = \{A \subseteq \mathbb{N} : A \) is a finite subset\}, asymptotically \( \mathcal{I} \)-statistical equivalent of order \( \alpha \) to multiple \( L \) coincides with asymptotically statistical equivalent of order \( \alpha \) to multiple \( L \). For an arbitrary ideal \( \mathcal{I} \) and for \( \alpha = 1 \) it coincides with asymptotically \( \mathcal{I} \)-statistical equivalent of multiple \( L \). When \( \mathcal{I} = \mathcal{I}_{\text{f}} \) and \( \alpha = 1 \) it becomes only asymptotically statistical equivalent of multiple \( L \), [10].

**Definition 10.** Let \( \theta \) be a lacunary sequence. A sequence \( x = (x_k) \) is said to be \( \mathcal{I} \)-lacunary statistically convergent of order \( \alpha \) to \( L \) or \( S_{\theta}(\mathcal{I})^\alpha \)-convergent to \( L \) if for any \( \varepsilon > 0 \) and \( \delta > 0 \)

\[
\{r \in \mathbb{N} : \frac{1}{h^\alpha_r} \sum_{k=r}^{\infty} \mathbb{1}_{\{|x_k - L| \geq \varepsilon\}} \geq \delta\} \in \mathcal{I}.
\]

In this case we write \( x_k \to L(S_{\theta}(\mathcal{I})^\alpha) \). The class of all \( \mathcal{I} \)-lacunary statistically convergent sequences of order \( \alpha \) will be denoted by \( S_{\theta}(\mathcal{I})^\alpha \).

We now have

**Definition 11.** Let \( \theta \) be a lacunary sequence; the two nonnegative sequences \( x = (x_k) \) and \( y = (y_k) \) are said to be asymptotically \( \mathcal{I} \)-lacunary statistical equivalent of order \( \alpha \) to multiple \( L \) provided that for any \( \varepsilon > 0 \) and \( \delta > 0 \)

\[
\{r \in \mathbb{N} : \frac{1}{h^\alpha_r} \sum_{k=r}^{\infty} \mathbb{1}_{\{|x_k - y_k - L| \geq \varepsilon\}} \geq \delta\} \in \mathcal{I},
\]

(\( \mathbb{1}_{\cdot} \) denotes by \( x - y \) and simply asymptotically \( \mathcal{I} \)-lacunary statistical equivalent of order \( \alpha \) if \( L = 1 \). Furthermore, let \( S^\alpha(\mathcal{I})^\theta \) denote the set of \( x \) and \( y \) such that \( x - y \).)

**Remark 2.** For \( \alpha = 1 \) the above definition coincides with asymptotically \( \mathcal{I} \)-lacunary statistical equivalent of multiple \( L \). Further it must be noted in this context that asymptotically \( \mathcal{I} \)-lacunary statistical equivalent of order \( \alpha \) to multiple \( L \) has not been studied till now. Obviously, if we take \( \mathcal{I} = \mathcal{I}_{\text{f}} \) asymptotically lacunary statistical equivalent of order \( \alpha \) to multiple \( L \) is a special case of asymptotically \( \mathcal{I} \)-lacunary statistical equivalent of order \( \alpha \) to multiple \( L \).

**Theorem 1.** Let \( 0 < \alpha \leq \beta \leq 1 \). Then \( S(\mathcal{I})^\alpha \subset S(\mathcal{I})^\beta \).

**Proof:** Let \( 0 < \alpha < \beta \leq 1 \). Then

\[
\frac{1}{n^\alpha} \sum_{k=n}^{\infty} \mathbb{1}_{\{|x_k - L| \geq \varepsilon\}} \leq \frac{1}{n^\beta} \sum_{k=n}^{\infty} \mathbb{1}_{\{|x_k - L| \geq \varepsilon\}}
\]

and so for any \( \delta > 0 \),
n we get to multiple it follows that is an ideal and . Then there exists . Define for sufficiently large , which belongs to if
\[ \delta > 0 \]
i.e.
\[ \delta \in (0, \infty) \]

Furthermore, let be a lacunary sequence; two number s.t.
\[ \theta \in (0, 1) \]

In particular , the following conditions remain true for and 
\[ \alpha \in (0, \infty) \]

Then for any 
\[ \delta > 0 \]

We now investigate the relationship between and 
\[ \delta > 0 \]

Then for any 
\[ \delta > 0 \]

This proves the result.

In order to establish that the inclusion \( N^L(\mathcal{I})^\sigma \subseteq S^L(\mathcal{I})^\sigma \) is proper, let \( \theta \) be given and define \( x_k \) to be 1,2,...\( \alpha \) integers in \( I_r \) and \( x_k = 0 \) otherwise for all \( r = 1,2,3,...,y_k = 1 \) for all \( k \). Then for any \( \varepsilon > 0 \),
\[ \frac{1}{h_r^\sigma} \left| \left\{ k \in I_r : \frac{x_k}{y_k} - L \geq \varepsilon \right\} \right| \leq \left\{ \frac{\sqrt{h_r^\sigma}}{h_r} \right\} + \frac{1}{4} \]
and for any \( \delta > 0 \) we get
\[ \left\{ r \in \mathbb{N} : \frac{1}{h_r^\sigma} \left| \left\{ k \in I_r : \frac{x_k}{y_k} - L \geq \varepsilon \right\} \right| \geq \delta \right\} \subseteq \{ r \in \mathbb{N} : \left\{ \frac{\sqrt{h_r^\sigma}}{h_r^\sigma} \right\} \geq \delta \} \]

Since the set on the right hand side is a finite set and so \( S^L(\mathcal{I})^\sigma \) belongs to \( \mathcal{I} \) it follows that \( x - y \). On the other hand
\[ \frac{1}{h_r^\sigma} \sum_{k \in I_r} \frac{x_k}{y_k} - 0 \geq \frac{1}{4} \left\{ \frac{\sqrt{h_r^\sigma}}{h_r^\sigma} \right\} + \frac{1}{2} \]
for some \( m \in \mathbb{N} \) which belongs to \( F(\mathcal{I}) \) since \( \mathcal{I} \) is admissible. So the following fails \( x - y \).

Remark 4. The following two conditions remain true for \( 0 < \alpha < 1 \) is not clear and we leave them as open problems.

\[ S^L(\mathcal{I})^\sigma = S^L(\mathcal{I})^\sigma \]

(2) \( x \in I_\sigma \) and \( x - y \) implies \( x - y \).

(3) \( S^L(\mathcal{I})^\sigma \cap I_\sigma = N^L(\mathcal{I})^\sigma \cap I_\sigma \).

We now investigate the relationship between \( s^L(\mathcal{I})^\sigma \) and \( s^L(\mathcal{I})^\sigma \) for sufficiently large \( \sigma \).

Then for any 
\[ \delta > 0 \]

This proves the result.

In order to establish that the inclusion \( N^L(\mathcal{I})^\sigma \subseteq S^L(\mathcal{I})^\sigma \) is proper, let \( \theta \) be given and define \( x_k \) to be 1,2,...\( \alpha \) integers in \( I_r \) and \( x_k = 0 \) otherwise for all \( r = 1,2,3,...,y_k = 1 \) for all \( k \). Then for any \( \varepsilon > 0 \),
\[ \frac{1}{h_r^\sigma} \left| \left\{ k \in I_r : \frac{x_k}{y_k} - L \geq \varepsilon \right\} \right| \leq \left\{ \frac{\sqrt{h_r^\sigma}}{h_r^\sigma} \right\} + \frac{1}{4} \]
and for any \( \delta > 0 \) we get
\[ \left\{ r \in \mathbb{N} : \frac{1}{h_r^\sigma} \left| \left\{ k \in I_r : \frac{x_k}{y_k} - L \geq \varepsilon \right\} \right| \geq \delta \right\} \subseteq \{ r \in \mathbb{N} : \left\{ \frac{\sqrt{h_r^\sigma}}{h_r^\sigma} \right\} \geq \delta \} \]

Since the set on the right hand side is a finite set and so \( S^L(\mathcal{I})^\sigma \) belongs to \( \mathcal{I} \) it follows that \( x - y \). On the other hand
\[ \frac{1}{h_r^\sigma} \sum_{k \in I_r} \frac{x_k}{y_k} - 0 \geq \frac{1}{4} \left\{ \frac{\sqrt{h_r^\sigma}}{h_r^\sigma} \right\} + \frac{1}{2} \]
for some \( m \in \mathbb{N} \) which belongs to \( F(\mathcal{I}) \) since \( \mathcal{I} \) is admissible. So the following fails \( x - y \).

Remark 4. The following two conditions remain true for \( 0 < \alpha < 1 \) is not clear and we leave them as open problems.

\[ S^L(\mathcal{I})^\sigma = S^L(\mathcal{I})^\sigma \]

(2) \( x \in I_\sigma \) and \( x - y \) implies \( x - y \).

(3) \( S^L(\mathcal{I})^\sigma \cap I_\sigma = N^L(\mathcal{I})^\sigma \cap I_\sigma \).

We now investigate the relationship between \( s^L(\mathcal{I})^\sigma \) and \( s^L(\mathcal{I})^\sigma \) for sufficiently large \( \sigma \).

Then for any 
\[ \delta > 0 \]

This proves the result.
\[ \{ r \in \mathbb{N} : \frac{1}{k_r^a} \left| \{ k \in I_r : \frac{x_k}{y_k} - L \geq \varepsilon \} \right| \geq \delta \} \subseteq \{ r \in \mathbb{N} : \frac{1}{k_r^a} \left| \{ k \leq k_r : \frac{x_k}{y_k} - L \geq \varepsilon \} \right| \geq \frac{\delta \sigma}{(1 + \sigma)} \in T. \]

This proves the result.

**Remark 5.** The converse of this result is not clear for \( \alpha < 1 \) and we leave it as an open problem.

For the next result we assume that the lacunary sequence \( \theta \) satisfies the condition that for any set \( C \in F(T) \),
\[ \bigcup \{ n : k_{r-1} < n < k_r, r \in C \} \subseteq T. \]

**Theorem 5.** For a lacunary sequence \( \theta \) satisfying the above condition,
\[ s^*_\theta([r])^\alpha \quad x - y \text{ implies } x - y \]
if \[ \sup_r \sum_{r=0}^{r-1} \frac{h_r^a}{(k_r^a)^\alpha} = B(\text{say}) < \infty. \]

**Proof:** Suppose that \( x - y \) and for \( \varepsilon, \delta, \delta_1 > 0 \) define the sets
\[ C = \{ r \in \mathbb{N} : \frac{1}{k_r^a} \left| \{ k \in I_r : \frac{x_k}{y_k} - L \geq \varepsilon \} \right| < \delta \} \]
and
\[ T = \{ n \in \mathbb{N} : \frac{1}{n^a} \left| \{ k \leq n : \frac{x_k}{y_k} - L \geq \varepsilon \} \right| < \delta_1 \}. \]

It is obvious from our assumption that \( C \in F(T) \), the filter associated with the ideal \( T \). Further observe that
\[ A_j = \frac{1}{k_r^a} \left| \{ k \in I_r : \frac{x_k}{y_k} - L \geq \varepsilon \} \right| < \delta \]
for all \( j \in C \). Let \( n \in \mathbb{N} \) be such that \( k_{r-1} < n < k_r \) for some \( r \in C \). Now
\[ \frac{1}{n^a} \left| \{ k \leq n : \frac{x_k}{y_k} - L \geq \varepsilon \} \right| \leq \frac{1}{k_{r-1}^a} \left| \{ k \leq k_r : \frac{x_k}{y_k} - L \geq \varepsilon \} \right| \]
\[ = \frac{1}{k_{r-1}^a} \left| \{ k \in I_r : \frac{x_k}{y_k} - L \geq \varepsilon \} \right| + \ldots + \frac{1}{k_r^a} \left| \{ k \in I_r : \frac{x_k}{y_k} - L \geq \varepsilon \} \right| \]
\[ = \frac{k_r^a}{k_{r-1}^a} A_1 + \frac{(k_r - k_{r-1})^a}{k_{r-1}^a} A_2 + \ldots + \frac{(k_n - k_{r-1})^a}{k_{r-1}^a} A_n \]
\[ \leq \sup_j A_j \sup_r \sum_{r=0}^{r-1} \frac{(k_r - k_{r-1})^a}{k_{r-1}^a} < B\delta. \]

Choosing \( \delta_1 = \frac{\delta}{B} \) and in view of the fact that
\[ \bigcup \{ n : k_{r-1} < n < k_r, r \in C \} \subseteq T \]
where \( C \in F(T) \) it follows from our assumption on \( \theta \) that the set \( T \) also belongs to \( F(T) \) and this completes the proof of the theorem.

**REFERENCES**


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