

Solution of an Optimal Control Problem with Mathcad

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Abstract — The paper deals with optimal control problems whose behavior is described by an elliptic equations with Bitsadze–Samarski nonlocal boundary conditions. The theorem about a necessary and sufficient optimality condition is given. The existence and uniqueness of a solution of the conjugate problem are proved. A numerical method of the solution of an optimal problem by means of the Mathcad package is presented.

Keywords — Elliptic equation, nonlocal boundary value problem, Bitsadze–Samarski problem, optimal control.

INTRODUCTION

Nonlocal boundary value problems are a very interesting generalization of classical problems and at the same time they are obtained in a natural manner when constructing mathematical models of real processes and phenomena in physics, engineering, sociology, ecology and so on [1]–[3]. The Bitsadze–Samarski nonlocal boundary value problem [4] arose in connection with mathematical modeling of processes occurring in plasma physics. Intensive studies of Bitsadze–Samarski nonlocal problems [5] and its various generalizations began in the 80s of the last century [4]–[6].

The present paper deals with optimal control problems whose behavior is described by Helmholtz equation with Bitsadze–Samarski nonlocal boundary conditions [7]. Necessary optimality conditions are established by using the approach worked out in [8]–[10] for controlled systems of general type. To investigate the conjugate problem we use the algorithm reducing nonlocal boundary value problems to a sequence of Dirichlet problems. Such a method makes it possible not only to solve the problem numerically, but also to prove the existence of its solution [5].

In the paper: the Bitsadze–Samarski boundary value problem for Helmholtz equations is considered. An optimal control problem is stated for a nonlocal boundary value problem with an integral quality test. A theorem about a necessary and sufficient optimality condition is formulated. The existence and uniqueness of a solution of the conjugate problem are proved. A numerical method is presented for solving an optimal problem by means of the Mathcad package.

I. STATEMENT OF THE OPTIMAL CONTROL PROBLEM

Let \bar{G} be the rectangle, $\bar{G} = [0,1] \times [0,1]$, Γ be the boundary of the domain G , $0 < x_0 < 1$, $\gamma_0 = \{(x_0, y) : 0 \leq y \leq 1\}$, $\gamma = \{(1, y) : 0 \leq y \leq 1\}$, $a(x, y), b(x, y), c(x, y), d(x, y) \in L_p(\bar{G})$, $p > 2$, $0 \leq q(x, y) \in L_\infty(\bar{G})$.

Let U be an arbitrary bounded set from R . Every function $\omega(x, y) : G \rightarrow U$ will be called a control. The set U is called the control domain. A function $\omega(x, y)$ is called an admissible control if $\omega(x, y) \in L_p(G)$, $p > 2$. The set of all admissible controls is denoted by Ω .

For each fixed $\omega(x, y) \in \Omega$ in the domain \bar{G} let us consider the following Bitsadze–Samarski boundary value problem for Helmholtz equations:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - q(x, y)u = a(x, y)\omega(x, y) + b(x, y), \quad (x, y) \in G, \quad (1.1)$$

$$u(x, y) = 0, \quad (x, y) \in \Gamma \setminus \gamma,$$

$$u(1, y) = \sigma u(x_0, y), \quad 0 \leq y \leq 1, \quad 0 < \sigma < 1.$$

The solution of the problem (1.1) exists, is unique and belongs to the space $W_2^2(G) \cap \dot{W}_2^1(G)$ [10].

We consider the functional

$$I(\omega) = \iint_G [c(x, y)u(x, y) + d(x, y)\omega(x, y)] dx dy \quad (1.2)$$

and state the following optimal control problem: Find a function $\omega_0(x, y) \in \Omega$, for which the solution of problem (1.2) gives functional (1.2) a minimal value. A function $\omega_0(x, y) \in \Omega$ will be called an optimal control, and the corresponding solution $u_0(x, y)$ an optimal solution.

Theorem. Let ψ_0 be the solution of the adjoint problem

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} - q(x, y)\psi = -c(x, y), \quad (x, y) \in G \setminus \gamma_0,$$

$$\psi(x, y) = 0, \quad (x, y) \in \Gamma, \quad (1.3)$$

$$\frac{\partial \psi(x_0^+, y)}{\partial x} - \frac{\partial \psi(x_0^-, y)}{\partial x} = \sigma \frac{\partial \psi(1, y)}{\partial x}, \quad 0 \leq y \leq 1.$$

Then for (u_0, ω_0) to be optimal it is necessary and sufficient that the principle of minimum

$$\inf_{\omega \in U} [d(x, y) + a(x, y)\psi_0(x, y)]\omega =$$

$$= [d(x, y) + a(x, y)\psi_0(x, y)]\omega_0$$

be fulfilled almost everywhere on G [11].

II. EXISTENCE AND UNIQUENESS OF A SOLUTION OF THE CONJUGATE PROBLEM

We write a solution of the conjugate problem (1.3) in the form $\psi = w + w^*$, where w^* is the solution of the Dirichlet problem

$$\frac{\partial^2 w^*}{\partial x^2} + \frac{\partial^2 w^*}{\partial y^2} - q(x, y)w^* = -c(x, y), \quad (x, y) \in G, \quad (2.1)$$

$$w^*(x, y) = 0, \quad (x, y) \in \Gamma,$$

and w is the solution of the nonclassical boundary value problem

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - q(x, y)w = 0, \quad (x, y) \in G \setminus \gamma_0,$$

$$w(x, y) = 0, \quad (x, y) \in \Gamma,$$

$$\frac{\partial w(x_0^+, y)}{\partial x} - \frac{\partial w(x_0^-, y)}{\partial x} =$$

$$= \sigma \frac{\partial w(1, y)}{\partial x} + \sigma \frac{\partial w^*(1, y)}{\partial x}, \quad 0 \leq y \leq 1. \quad (2.2)$$

As is known [12], problem (2.1) has a unique solution that belongs to the space $W_2^2(G) \cap \overset{\circ}{W}_2^1(G)$. Therefore it remains for us to investigate problem (2.1). To this end, we consider the iteration process

$$\frac{\partial^2 w^{k+1}}{\partial x^2} + \frac{\partial^2 w^{k+1}}{\partial y^2} - q(x, y)w^{k+1} = 0, \quad (x, y) \in G \setminus \gamma_0,$$

$$w^{k+1}(x, y) = 0, \quad (x, y) \in \Gamma,$$

$$\frac{\partial w^{k+1}(x_0^+, y)}{\partial x} - \frac{\partial w^{k+1}(x_0^-, y)}{\partial x} =$$

$$= \sigma \frac{\partial w^k(1, y)}{\partial x} + \varphi(y), \quad 0 \leq y \leq 1, \quad k = 0, 1, 2, \dots,$$

where $\varphi(y) = \sigma \frac{\partial w^*(1, y)}{\partial x}$, $w^0(x, y)$ is the initial approximation which can be assumed to be equal to zero. Let $z^{(k)}(x, y) = w^{(k+1)}(x, y) - w^{(k)}(x, y)$, then from equalities (2.3) we obtain

$$\frac{\partial^2 z^k}{\partial x^2} + \frac{\partial^2 z^k}{\partial y^2} - q(x, y)z^k = 0, \quad (x, y) \in G \setminus \gamma_0,$$

$$z^k(x, y) = 0, \quad (x, y) \in \Gamma,$$

$$\frac{\partial z^k(x_0^+, y)}{\partial x} - \frac{\partial z^k(x_0^-, y)}{\partial x} =$$

$$= \sigma \frac{\partial z^{k-1}(1, y)}{\partial x}, \quad 0 \leq y \leq 1, \quad k = 1, 2, 3, \dots \quad (2.4)$$

In the Space $W_2^2(G \setminus \gamma_0) \cap \overset{\circ}{W}_2^1(G)$ problem (2.4) is equivalent to the following problem [13], [14]

$$\frac{\partial^2 z^k}{\partial x^2} + \frac{\partial^2 z^k}{\partial y^2} - q(x, y)z^k =$$

$$= -\sigma \delta(x_0 - x) \frac{\partial z^{k-1}(1, y)}{\partial x}, \quad (x, y) \in G, \quad (2.5)$$

$$z^k(x, y) = 0, \quad (x, y) \in \Gamma, \quad k = 1, 2, 3, \dots,$$

where $\delta(x_0 - x)$ is the Dirac function. Let us introduce the notation

$$f(x, y) = -\sigma \delta(x_0 - x) \frac{\partial z^{(k-1)}(1, y)}{\partial x}.$$

Since $f(x, y) \in W_2^{-1}(G)$, the solution of problem (2.5) exists, is unique and belongs to the space $\overset{\circ}{W}_2^1(G)$ [12]. Moreover, if $G(x, y, \xi, \eta)$ is the Green's function of problem (2.5), then the solution can be represented as follows

$$z^{(k)}(x, y) =$$

$$= \iint_G \sigma G(x, y, \xi, \eta) \delta(x_0 - \xi) \frac{\partial z^{(k-1)}(1, \eta)}{\partial x} d\xi d\eta =$$

$$= \int_0^1 \sigma G(x, y, x_0, \eta) \frac{\partial z^{(k-1)}(1, \eta)}{\partial x} d\eta. \quad (2.6)$$

Furthermore, taking the property of the Green's function into account [13], [14], from equality (2.6) it follows that

$z^{(k)}(x, y) \in W_2^2(G \setminus \gamma_0) \cap \overset{\circ}{W}_2^1(G)$, hence we define the trace of the function $\frac{\partial}{\partial x}(z^{(k)}(1, y))$ which belongs to the space

$W_2^{1/2}(0, 1)$. Thus, using (2.6) we can write

$$\frac{\partial z^{(k)}(1, y)}{\partial x} = \int_0^1 \sigma \frac{\partial G(1, y, x_0, \eta)}{\partial x} \cdot \frac{\partial z^{(k-1)}(1, \eta)}{\partial x} d\eta. \quad (2.7)$$

Next, using the Cauchy–Bunyakovsky inequality, from equality (2.7) we obtain

$$\int_0^1 \left[\frac{\partial z^{(k)}(1, y)}{\partial x} \right]^2 dy \leq$$

$$\leq \sigma^2 \int_0^1 \int_0^1 \left[\frac{\partial G(1, y, x_0, \eta)}{\partial x} \right]^2 dx dy \cdot \int_0^1 \left[\frac{\partial z^{(k-1)}(1, \eta)}{\partial x} \right]^2 d\eta. \quad (2.8)$$

Denote

$$\theta = \sigma \left\| \frac{\partial G(1, y, x_0, \eta)}{\partial x} \right\|_{L_2((0,1) \times [0,1])},$$

then from (2.8) we obtain the estimate

$$\left\| \frac{\partial z^{(k)}(1, y)}{\partial x} \right\|_{L_2[0,1]} \leq \theta \left\| \frac{\partial z^{(k-1)}(1, y)}{\partial x} \right\|_{L_2[0,1]} \quad (2.9)$$

Let $\sigma < 1 / \left\| \frac{\partial}{\partial x} (G(1, y, x_0, \eta)) \right\|$, then $\theta < 1$ and estimate

(2.9) implies that the series $\sum_{k=1}^{\infty} \frac{\partial}{\partial x} (z^{(k)}(1, y))$ converges. This

means that the sequence $\left\{ \frac{\partial w^{(k)}(1, y)}{\partial x} \right\}$ converges, where

$$\frac{\partial w^{(k)}(1, y)}{\partial x} = \sum_{i=0}^{k-1} \left[\frac{\partial w^{(i+1)}(1, y)}{\partial x} - \frac{\partial w^{(i)}(1, y)}{\partial x} \right] + \frac{\partial w^{(0)}(1, y)}{\partial x}$$

in the norm of the space $L_2[0,1]$. Therefore equality (2.6)

implies the convergence of the series $\sum_{k=1}^{\infty} z^{(k)}(x, y)$, and thereby

the convergence of the iteration sequence $\{w^{(k)}(x, y)\}$:

$$\lim_{k \rightarrow \infty} w^{(k)}(x, y) = w(x, y),$$

where $w(x, y) \in W_2^2(G \setminus \gamma_0) \cap \overset{\circ}{W}_2^1(G)$. Further it is not difficult to prove that the function $w(x, y)$ is the solution of problem (2.2).

Let us prove the uniqueness of the solution. Assume that $w_1(x, y)$ and $w_2(x, y)$ are two generalized solutions of problem (2.2). Then their difference $z(x, y) = w_1(x, y) - w_2(x, y)$ is the solution of the problem

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} - q(x, y)z = 0, \quad (x, y) \in G \setminus \gamma_0, \\ z(x, y) = 0, \quad (x, y) \in \Gamma, \quad (2.10)$$

$$\frac{\partial z(x_0^+, y)}{\partial x} - \frac{\partial z(x_0^-, y)}{\partial x} = \sigma \frac{\partial z(1, y)}{\partial x}, \quad 0 \leq y \leq 1.$$

By the same reasoning as above we obtain $z(x, y) = 0$. We have thereby proved the existence and uniqueness of the solution of problem (1.3).

III. ALGORITHM OF THE SOLUTION OF AN OPTIMAL CONTROL PROBLEM

The scheme of the solution of an optimal control problem is as follows:

1. to find a solution $\psi_0(x, y)$, we first solve the adjoint problem (1.3);
2. using the function $\psi_0(x, y)$ from (1.4), we construct the optimal control $\omega_0(x, y)$;
3. to find an optimal solution $u_0(x, y)$, we solve problem (1.1).

As we have shown, a solution of the adjoint problem can be written in the form $\psi = w + w^*$, where w^* is the solution of

the Dirichlet problem (2.1), and w is the solution of the nonclassical boundary value problem (2.2). For the solution of problem (2.2) we consider the iteration process (2.3). Problem (2.3) is equivalent to the following problem

$$\frac{\partial^2 w^{k+1}}{\partial x^2} + \frac{\partial^2 w^{k+1}}{\partial y^2} - q(x, y)w^{k+1} = \\ = \delta(x_0 - x) \left(\sigma \frac{\partial w^k(1, y)}{\partial x} + \varphi(y) \right), \quad (x, y) \in G, \quad (3.1)$$

$$w^{k+1}(x, y) = 0, \quad (x, y) \in \Gamma, \quad k = 0, 1, 2, \dots,$$

where $\varphi(y) = \sigma \frac{\partial w^*(1, y)}{\partial x}$, $w^0(x, y)$ is the initial approximation.

For the solution of problem (1.1) we consider the iteration process

$$\frac{\partial^2 u^{k+1}}{\partial x^2} + \frac{\partial^2 u^{k+1}}{\partial y^2} - q(x, y)u^{k+1} = \\ = a(x, y)u + b(x, y), \quad (x, y) \in G, \\ u^{k+1}(x, y) = 0, \quad (x, y) \in \Gamma \setminus \gamma, \quad (3.2)$$

$$u^{k+1}(1, y) = \sigma u^k(x_0, y), \quad 0 \leq y \leq 1, \quad 0 < \sigma < 1,$$

$$k = 0, 1, 2, \dots$$

IV. NUMERICAL REALIZATION OF THE PROBLEM BY MEANS OF THE MATHCAD

To obtain a numerical solution of problems (3.1) and (3.2), at each iteration step we use the built-in function **relax(a, b, c, d, e, f, u, rjac)** in the Mathcad [15], [16].

The function **relax** returns the square matrix, where the position of an element in the matrix corresponds to its position inside the square domain, while the value corresponds to an approximate solution at this point.

The arguments of the function **relax** are as follows:

a, b, c, d, e are square matrices of one and the same size, containing the coefficients of a differential equation. In particular, for a Helmholtz equation the coefficients are $a_{i,j} = b_{i,j} = c_{i,j} = d_{i,j} = 1$, $e_{i,j} = -4 - q_{i,j}$, where $q_{i,j}$ are the values of $q(x, y)$ in the respective node inside the square domain;

f is a square matrix containing the values of the right-hand part of the equation in the respective node inside the square domain;

u is a square matrix containing the boundary values of the function at the domain edges, and also the initial approximation of the solution in the interior nodes of the square domain;

rjac is the parameter controlling the relaxation process convergence. It may vary in the range from 0 to 1, but an optimal value depends on the particulars of the problem.

As an example let us consider problems (1.1) and (1.3), where $q(x, y) = x + y$, $a(x, y) = 1$, $b(x, y) = 0$, $c(x, y) = 1$, $d(x, y) = 1$, $\omega(x, y) = 1$, $x_0 = 1/2$.

The results of the numerical solution of problems (1.1) and

(1.3) are graphically shown in Fig. 1. and Fig. 2.

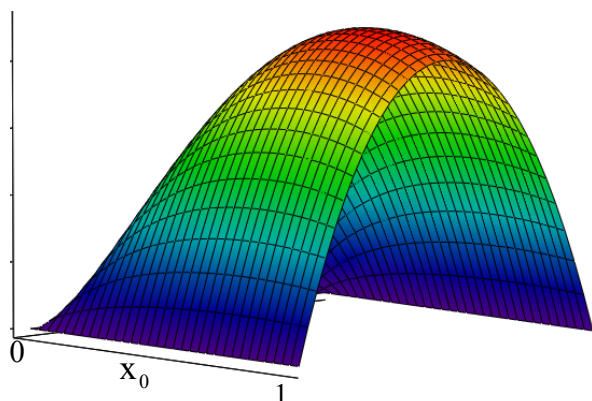


Fig. 1. Solution Bitsadze–Samarski problem

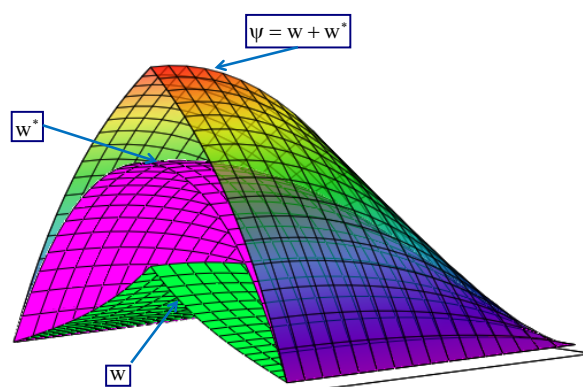


Fig. 2. Solution adjoint problem

REFERENCES

- [1] V. V. Shelukhin, A non-local in time model for radionuclides propagation in Stokes fluid. *Dinamika Sploshn. Sredy* No. 107 (1993), 180–193, 203, 207.
- [2] C. V. Pao, Reaction diffusion equations with nonlocal boundary and nonlocal initial conditions. *J. Math. Anal. Appl.* 195 (1995), No. 3, 702–718.
- [3] E. Obolashvili, Nonlocal problems for some partial differential equations. *Appl. Anal.* 45 (1992), No. 1-4, 269–280.
- [4] A. V. Bitsadze and A. A. Samarskii, On some simple generalizations of linear elliptic boundary problems. (Russian) *Dokl. Akad. Nauk SSSR* 185 (1969), 739-740; English transl.: *Sov. Math., Dokl.* 10 (1969), 398–400.
- [5] D. G. Gordeziani, On the methods of solution of one class of nonlocal boundary value problems. Tbilisi State University Press, Tbilisi, 1986.
- [6] D. V. Kapanadze, On a nonlocal Bitsadze-Samarski boundary value problem. (Russian) *Differentsial'nye Uravneniya* 23 (1987), No. 3, 543–545, 552.
- [7] D. Devadze, V. Beridze. A Control Optimal Problem for Helmholtz Equations with Bitsadze–Samarski Boundary Conditions. *Proceedings of A. Razmadze Mathematical Institute.* 2013. Vol. 161. pp. 47-53.

- [8] D. Sh. Devadze and V. Sh. Beridze, Optimality conditions for quasilinear differential equations with nonlocal boundary conditions. *UMN* 68 (2013), No. 4. pp 179-180.
- [9] V. I. Plotnikov, Necessary and sufficient conditions for optimality and conditions for uniqueness of the optimizing functions for control systems of general form. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* 36 (1972), 652–679.
- [10] G. Berikelashvili. On the Solvability of a Nonlocal Boundary Value Problem in the Weighted Sobolev Spaces. *Proceedings of A. Razmadze Mathematical Institute* 119 (1999), 3-11
- [11] G. V. Meladze, T. S. Tsutsunava, and D. Sh. Devadze, An optimal control problem for quasi-linear differential equations of first order on the plane with nonlocal boundary conditions. Tbilisi State University Press, Tbilisi, 1987, deposited at the Georgian Institute of Technical and Scientific Information, 25.12.87, No. 372, Г87, 61 p.
- [12] O. A. Ladizhenskaya and N. N. Uraltseva, Linear and quasilinear equations of elliptic type. (Russian) Nauka, Moscow, 1973.
- [13] V. S. Vladimirov, The equations of mathematical physics. (Russian) Fourth edition. “Nauka”, Moscow, 1981.
- [14] S. L. Sobolev, The Equations of Mathematical Physics. (Russian) Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow–Leningrad, 1950.
- [15] D. V. Kiryanov, *Mathcad 14*. BHV-Petersburg, St. Petersburg, 2007.
- [16] M. Herhager and J. Partoll, *Mathcad 2000: The Complete Guide*: Trans. with Ger.-K. Publishing Group, BHV, 2000.

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