Generalization of Fourier transform and Weyl calculus

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Abstract: **- In this paper, a surjective morphism of the topological groups from the real line R to the p-curve Cp is introduced, this function maps from the real line to the p-curve on the complex and when $p=2$ then coincide with a classical exponent. The properties of p-Fourier transform is studied. The generalization of the Weyl functional calculus is considered.**

Key-Words: **- General periodic function, Fourier analysis, p-circle, spectral theory, oscillation.**

I. INTRODUCTION

The paramount example of a linear isomorphism from one Hilbert space $H = L^2(R, dx)$ to another Hilbert space $\hat{H} = L^2(R, d\lambda)$ is the Fourier transform, which transforms a complex function ψ into a different complex function $\hat{\psi}$. If we assume that $\phi(\lambda)$ is a real function of the real argument then we can define a family of operators $\exp(-2\pi i t\phi(\lambda))(\lambda)$, that operator family constitutes a unitary group. The possibility of this construction is rendered by the exponentiation identity for the Fourier operator.

The importance of the Fourier transformation is due to its wide applications in modern physics, especially, which utilize quantum approaches to the description of natural processes, and information science, it is a fundamental tool of signal processing.

In the present paper, we make an attempt to generalize the theory of the Fourier functional calculus by introducing the pair of circular functions $pcs(\theta)$ and $psn(\theta)$, and extending the definitions of the Weyl theory.

A curved line given by the equation $|x|^p + |y|^p = 1$ on R^2-plane is called a p-curve and denoted by Cp. Let us denote the length of p-curve by l_p. We introduce a pair of C^1-smooth functions $pcs(\theta)$ and $psn(\theta)$ of the real argument $\theta \in [0, l_p]$ defined as

$$pcs(\theta) = x \quad \text{for all} \quad \theta \in R \tag{1}$$

and

$$psn(\theta) = y \quad \text{for all} \quad \theta \in R, \tag{2}$$

where coordinates x and y belongs to p-curve, i.e. bound by the equation $|x|^p + |y|^p = 1$, so that

$$psn(0) = pcs\left(\frac{l_p}{4}\right) = 0 \text{ and}$$

$$pcs(0) = psn\left(\frac{l_p}{4}\right) = 1, \text{ and}$$

$$|psn(\theta)|^p + |pcs(\theta)|^p = 1 \quad \text{for all} \quad \theta \in R. \tag{3}$$

These functions satisfy the integral identity

$$psn(\theta)\, pcs(\theta) = \\ = \int \left((pcs(\theta))^p - (psn(\theta))^p \right) d\theta. \tag{4}$$

II. p-FOURIER TRANSFORM

Assume $f \in L^p\left[0, l_p\right]$ and let us write a Fourier-type series with appropriate weights on the interval $\left[0, l_p\right]$ as

$$f(x) = a_0 + \\ + \sum_{m=1,2,\dots} \left(a_m pcs(mx) + b_m psn(mx)\right), \tag{5}$$

with some real coefficients

$a_0, a_1, b_1, \dots, a_m, b_m, \dots$

By usual means. integrating the identity (3) over the period l_p, we obtain

$$\int_0^{l_p} \left|pcs(\theta)\right|^p d\theta = \int_0^{l_p} \left|psn(\theta)\right|^p d\theta = \frac{l_p}{2} \tag{6}$$

and

$$a_0 = \frac{1}{l_p} \int_0^{l_p} f(x)\, dx. \tag{7}$$

Next, we have

$$a_m = \frac{2}{l_p} \int_0^{l_p} f(x)\, pcs(mx) \left|pcs(mx)\right|^{p-2} dx \tag{8}$$

and

$$b_m = \frac{2}{l_p} \int_0^{l_p} f(x)\, psn(mx) \left|psn(mx)\right|^{p-2} dx. \tag{9}$$

Thus, we obtain the mapping of the functions $f \in L^p\left[0, l_p\right]$ in the set of the infinite series according to the formula

$$f(x) = \frac{1}{l_p} \int_0^{l_p} f(y)\, dy + \\ \sum_m \int_0^{l_p} \begin{pmatrix} f(y)\, pcs(my) \left|pcs(my)\right|^{p-2} pcs(mx) + \\ + f(y)\, psn(my) \left|psn(my)\right|^{p-2} psn(mx) \end{pmatrix} \frac{2dy}{l_p}. \tag{10}$$

Statement (analogous Riemannian theorem) 1. Assuming g is an integrable function over an arbitrary interval $[a, b] \subset R$ then

$$\lim_{m \to \infty} \int_a^b g(x)\, psn(mx) \left|psn(mx)\right|^{p-2} dx = 0 \tag{11}$$

and

$$\lim_{m \to \infty} \int_a^b g(x)\, pcs(mx) \left|pcs(mx)\right|^{p-2} dx = 0. \tag{12}$$

Theorem (adjoint) 2. Let g be an integrable function over an arbitrary interval $[a, b] \subset R$ then there are

$$\lim_{m \to \infty} \int_a^b g(x)\, psn(mx)\, dx = 0 \tag{13}$$

and

$$\lim_{m \to \infty} \int_a^b g(x)\, pcs(mx)\, dx = 0. \tag{14}$$

Adjoint series

Assume $f \in L^p$ then $f\left|f\right|^{p-2} \in L^{\frac{p}{p-1}}$ and we can write

$$f(x)\left|f(x)\right|^{p-2} = \\ = \tilde{a}_0 + \sum_{m=1,2,\dots} \begin{pmatrix} \tilde{a}_m pcs(mx) \left|pcs(mx)\right|^{p-2} + \\ + \tilde{b}_m psn(mx) \left|psn(mx)\right|^{p-2} \end{pmatrix}, \tag{15}$$

where $\tilde{a}_0, \tilde{a}_1, \tilde{b}_1, \dots, \tilde{a}_m, \tilde{b}_m, \dots$ defined as follows

$$\tilde{a}_0 = \frac{1}{l_p} \int_0^{l_p} f(x)\left|f(x)\right|^{p-2} dx, \tag{16}$$

$$\tilde{a}_m = \frac{2}{l_p} \int_0^{l_p} f(x)\left|f(x)\right|^{p-2} pcs(mx)\, dx \tag{17}$$

and

$$\tilde{b}_m = \frac{2}{l_p} \int_0^{l_p} f(x)\left|f(x)\right|^{p-2} psn(mx)\, dx. \tag{18}$$

III. THE MORPHISM FROM THE REAL LINE TO THE COMPLEX PLANE $Epp: R \to Cp$

We introduce a function $Epp: R \to Cp$, which maps from the real line to the p-curve on the complex plane as follows

$$Epp(i\theta) = pcs(\theta) + i\, psn(\theta), \quad \theta \in R \tag{19}$$

and dual function

$$Epq(i\theta) = pcs(\theta) + i\, psn(\theta), \quad \theta \in R, \quad p = q, \tag{20}$$

assume that p is renaming q. The function $Epp: R \to Cp$ is a surjective morphism of the topological groups from the real line R to the p-curve Cp and covering the space of the p-curve Cp. In case $p = 2$, the function Epp is a classical exponent on the complex plane of the imaginary argument.

From formula (19), we have

$$pcs(\theta)=\frac{1}{2}\left(Epp\,(i\,\theta)+Epp\,(-i\,\theta)\right),\quad \theta\in R$$

and

$$psn(\theta)=\frac{1}{2i}\left(Epp\,(i\,\theta)-Epp\,(-i\,\theta)\right),\quad \theta\in R\,.$$

We introduce an integral transformation Tp of a function $f\in L^p\bigcap L^q$ in the form

$$^{p}\hat{f}(\lambda)=\int_{-\infty}^{\infty}Epp\left(-l_p i\lambda\cdot x\right)f(x)\,dx=Tp(f)(\lambda) \tag{21}$$

where l_p is a length of the p-curve Cp.

This integral transformation Tp is a linear mapping relative to the function f and in case $p=2$ coincides with the Fourier transformation.

If $p=2$ then the integral transformation of function g

$$\int_{-\infty}^{\infty}Epp\left(l_p i\lambda\cdot x\right)g(\lambda)\,d\lambda=Rp(g)(x) \tag{22}$$

coincides with the inverse Fourier transform, in the general case it is not necessarily true since the dual structure does not coincide with the natural complex structure, the inverse transform is not always given by formula (22).

We define the inverses integral transformation Tp^{-1} of a function $^{p}\hat{f}(\lambda)$ as

$$f(x)=Tp^{-1}\left({}^{p}\hat{f}\right)(x) \tag{23}$$

for all transforms $^{p}\hat{f}(\lambda)$.

So, we introduce two types of mappings: first is an analog of the Fourier transform Tp and its inverse Tp^{-1}, second is an analog of the inverse Fourier transform Rp and we can easily define its inverse Rp^{-1}. These morphisms do not have the structure of the group except for $p=2$.

IV. GENERALIZATION OF THE WIGNER FUNCTION

Let functions $\psi\in L^p\left(R^n\right)$ and $\varphi\in L^q\left(R^n\right)$ then we introduce a general Wigner function $W_\eta(\psi,\varphi)(x,p)$ as any quasi-probability distribution, which satisfies the following conditions:

1. $\int_{R^n}W_\eta(\psi,\varphi)(x,p)\,dp=\psi(x)\bar{\varphi}(x)$;
2. $\int_{R^n}W_\eta(\psi,\varphi)(x,p)\,dx=Tp(\psi(p))\overline{Tp}(\varphi(p))$.

As a consequence of the first condition, we have

$$\int_{R^{2n}}W_\eta(\psi,\varphi)(x,p)\,dp\,dx=\left\langle\psi(x)\bar{\varphi}(x)\right\rangle_x .$$

For a pair of functions $\psi\in L^p\left(R^n\right)$ and $\varphi\in L^q\left(R^n\right)$ such that $\langle\psi\,|\,\varphi\rangle\neq 0$, we define a density ρ in the point (x,p) by

$$\rho_{\psi,\varphi}(x,p)=\overline{\rho_{\psi,\varphi}(x,p)}=\frac{W_\eta(\psi,\varphi)(x,p)}{\langle\psi\,|\,\varphi\rangle}.$$

The probability density function is a homogeneous function of degree one so that $\rho_{\lambda\psi,\lambda\varphi}(x,p)=\rho_{\psi,\varphi}(x,p)$ for all complex $\lambda\neq 0$.

Let us introduce the generalization of the Weyl quantization by

$$\left(\Im_\sigma\psi\right)(\lambda)=\int_{R^n}Epp\left(l_p i\,\sigma(\lambda,x)\right)\psi(x)\,dx\,,$$

where σ is a symplectic form.

We define an operator

$$V(\lambda)=Epp\left(-l_p i\,\sigma\left((\lambda,x),(Q,P)\right)\right),$$

where Q is position operators and P is a momentum.

The Weyl quantization $Dp(\psi)(\phi)$ is defined by

$$Dp(\psi)(\phi)=\left\langle\left(\Im_\sigma\psi\right)(\cdot)V(\cdot)\phi(\cdot)\right\rangle$$

for any test function ϕ.

We estimate $\|Dp(\psi)(\phi)\| \leq \|\Im_\sigma \psi\|_p \|\phi\|$. Similarly, to the classical case, the new Weyl quantization is a linear mapping so that

$$Dp(\alpha\psi + \beta\varphi) = \alpha Dp(\psi) + \beta Dp(\varphi)$$

holds for all complex numbers α, β.

Definition. *The Schwartz space is a space of all functions such that*

$$S(R^n) = \\ = \begin{Bmatrix} \psi \in C^\infty(R^n) & : & \sup\limits_{x \in R^n} \left| x^a \partial_x^\alpha \psi(x) \right| < \infty \\ \forall \alpha, a \in N^n \bigcup \{0\} & & \end{Bmatrix}.$$

Now, let us consider a case when $Epp = Exp$. The exponent function satisfies the characteristic identity $Exp(a+b) = Exp(a)Exp(b)$ so the Weyl product has the property

$$Dp(\psi \# \varphi) = Dp(\psi) Dp(\varphi)$$

for some function ψ, φ. The symbol # denotes a non-commutative product (often called Weyl product) so that $Dp(\psi \# \varphi) = Dp(\psi) \cdot Dp(\varphi)$ for some functions.

Let us assume K_A and K_B are kernels for the integral operators A and B respectively. So, we have

$$Dp\left(Dp^{-1}(A)\phi\right)(x) = \\ = \int_{R^{2n}} \begin{pmatrix} \exp(-2\pi i(z-x)p)\varepsilon^n \\ W_\eta(K_A)\left(\frac{1}{2}(x+z, \varepsilon p)\right)\phi(z) \end{pmatrix} dpdz = \\ = \varepsilon^n \int_{R^{3n}} \begin{pmatrix} \exp(-2\pi i(z-x+y)p)\phi(z) \times \\ K_A\left(\frac{1}{2}(z+x+y), \frac{1}{2}(z+x-y)\right) \end{pmatrix} dpdzdy,$$

we take $Dp(\psi) = A$ then $\psi = Dp^{-1}(A)$ and calculate

$$K_A\left(x + \frac{\varepsilon}{2}z, x - \frac{\varepsilon}{2}z\right) = \varepsilon^{-n}\left(F^{-1}\psi\right)(x, z),$$

thus

$$Dp^{-1}\left(Dp(\psi)\right)(x, p) = \psi(x, p).$$

Generally speaking, the product $K_A \square K_B \in S(R^n \times R^n)$ does not commute. So, we obtain the following lemma.

Lemma 1. ***Let K_A be a kernel of an operator $A \in BL\left(L^2(R^n), L^2(R^n)\right)$. Then the mapping Dp^{-1} is an inverse to Weyl quantization so that $Dp^{-1}A = \varepsilon^n W(K_A)$ and $A = Dp\left(\varepsilon^n W(K_A)\right)$; the Weyl kernel is given by***

$$K_\psi = \int_{R^n} \exp(-2\pi i(z-x)p)\psi\left(\frac{1}{2}(x+z, \varepsilon p)\right)dp = \\ = \varepsilon^n \left(F(\psi)\right)\left(\frac{1}{2}\left(x+z, \frac{z-x}{\varepsilon}\right)\right),$$

then $\begin{matrix} Dp^{-1}\left(Dp(\psi)\right)(x, p) = \\ = \varepsilon^n \left(W(K_\psi)\right)(x, p) = \psi(x, p) \end{matrix}$ ***holds for*** $\psi \in L^2(R^n)$.

Lemma 2. ***Let K_A and K_B be integral kernels of the operators A and B respectively. Then the product $(K_A \square K_B)(x, z) = \langle K_A(x, \cdot) K_B(\cdot, z) \rangle$ is correctly defined and is a kernel of the operator; in other words $\bullet : S(R^n \times R^n) \times S(R^n \times R^n) \to S(R^n \times R^n)$.***

Proof. Let us denote the multi-indices by $a, \alpha, b, \beta \in N_0^n$ then we estimate

$$\left| x^a z^b \partial_x^\alpha \partial_z^\beta (K_A \square K_B)(x, z) \right| = \\ = \left| x^a z^b \partial_x^\alpha \partial_z^\beta \langle K_A(x, \cdot) K_B(\cdot, z) \rangle \right| \leq \\ \leq \left\langle \left| x^a z^b \partial_x^\alpha \partial_z^\beta K_A(x, \cdot) K_B(\cdot, z) \right| \right\rangle = \\ = \left\| x^a z^b \partial_x^\alpha \partial_z^\beta K_A(x, \cdot) K_B(\cdot, z) \right\|_{L^1} \leq \\ \leq Const1 \sup_{\cdot \in R^n} \left| x^a z^b \partial_x^\alpha \partial_z^\beta K_A(x, \cdot) K_B(\cdot, z) \right| + \\ + Const2 \max_{|c| = 2n} \sup_{\cdot \in R^n} \left| x^a z^b \partial_x^\alpha \partial_z^\beta K_A(x, \cdot) K_B(\cdot, z) \right| \leq \\ \leq Const1 \left\| x^a z^b \partial_x^\alpha \partial_z^\beta K_A(x, \cdot) K_B(\cdot, z) \right\|_{00} + \\ + Const2 \max_{|c| = 2n} \left\| x^a z^b \partial_x^\alpha \partial_z^\beta K_A(x, \cdot) K_B(\cdot, z) \right\|_{c0}.$$

Next, we exchange the order of the supremum and integration and obtain

$$\sup_{x,\,z\in R^n}\left\langle\left|x^a z^b \partial_x^\alpha \partial_z^\beta K_A(x,\cdot)K_B(\cdot,z)\right|\right\rangle \le$$

$$\le\left\langle \sup_{x,\,z\in R^n}\left|x^a z^b \partial_x^\alpha \partial_z^\beta K_A(x,\cdot)K_B(\cdot,z)\right|\right\rangle =$$

$$=\left\|\sup_{x,\,z\in R^n}\left|x^a z^b \partial_x^\alpha \partial_z^\beta K_A(x,\cdot)K_B(\cdot,z)\right|\right\|_{L^1}$$

so, we have

$$\left\|\sup_{x,\,z\in R^n}\left|x^a z^b \partial_x^\alpha \partial_z^\beta K_A(x,\cdot)K_B(\cdot,z)\right|\right\|_{L^1} \le$$

$$\le C1 \sup_{x,\,z\in R^n}\sup_{\cdot\in R^n}\left|x^a z^b \partial_x^\alpha \partial_z^\beta K_A(x,\cdot)K_B(\cdot,z)\right| +$$

$$+C2\max_{|c|=2n}\sup_{x,\,z\in R^n}\sup_{\cdot\in R^n}\left|x^a z^b \partial_x^\alpha \partial_z^\beta K_A(x,\cdot)K_B(\cdot,z)\right|<\infty,$$

thus, we obtain $K_A \square K_B \in S\left(R^n\times R^n\right)$.

For the Weyl system, we can formulate the following Weyl quantization theorem.

Theorem. ***Let functions $\psi,\varphi\in S\left(R^{2n}\right)$ then the function $\psi\#\varphi\in S\left(R^{2n}\right)$ and such that satisfies the equality***

$$Dp(\psi\#\varphi)=Dp(\psi)Dp(\varphi),$$

where

$$(\psi\#\varphi)(x,p)=$$

$$\left\langle\left\langle \begin{array}{l}\exp\left(2\pi i\sigma\left((x,p),(z+\tilde{z},\eta+\tilde{\eta})\right)\right)\\ \exp\left(2\pi\frac{i\varepsilon}{2}\sigma\left((z,\eta),(\tilde{z},\tilde{\eta})\right)\right)\times\\ (F_\sigma\psi)(z,\eta)(F_\sigma\varphi)(\tilde{z},\tilde{\eta})\end{array}\right\rangle_{(z,\eta)}\right\rangle_{(\tilde{z},\tilde{\eta})}$$

$$\left\langle\left\langle \begin{array}{l}\exp\left(\dfrac{4\pi i\sigma\left((x,p)-(z,\eta),(x,p)-(\tilde{z},\tilde{\eta})\right)}{\varepsilon}\right)\\ \psi(z,\eta)\varphi(\tilde{z},\tilde{\eta})\end{array}\right\rangle\right\rangle$$

Proof. Assume $\psi,\varphi\in S\left(R^{2n}\right)$ and employ the definition of Dp, we have

$$\left(Dp(\psi)Dp(\varphi)\right)=$$

$$=\left\langle\left\langle \begin{array}{l}(F_\sigma\psi)(z,\eta)(F_\sigma\varphi)(\tilde{z},\tilde{\eta})\times\\ W(z,\eta)W(\tilde{z},\tilde{\eta})\end{array}\right\rangle_{(z,\eta)}\right\rangle_{(\tilde{z},\tilde{\eta})}=$$

$$\left\langle\left\langle \begin{array}{l}\exp\left(-2\pi\frac{2i}{\varepsilon}\sigma\left((z,\eta),(\tilde{z},\tilde{\eta})-(z,\eta)\right)\right)\times\\ (F_\sigma\psi)(z,\eta)(F_\sigma\varphi)\left((z,\eta)-(\tilde{z},\tilde{\eta})\right)\times\\ W(\tilde{z},\tilde{\eta})\end{array}\right\rangle\right\rangle.$$

Now, we are going to establish that $\psi\#\varphi\in S\left(R^{2n}\right)$

$$(\psi\#\varphi)(x,p)=$$

$$=F_\sigma\left(\left\langle \begin{array}{l}\exp\left(2\pi\frac{2i}{\varepsilon}\sigma\left((z,\eta),\cdot-(z,\eta)\right)\right)\\ (F_\sigma\psi)(z,\eta)(F_\sigma\varphi)\left(\cdot-(z,\eta)\right)\end{array}\right\rangle_{(z,\eta)}\right)(x,p)$$

$$=\left\langle\left\langle \begin{array}{l}\exp\left(2\pi i\sigma\left((x,p),(z,\eta)\right)\right)\\ \exp\left(2\pi\frac{i\varepsilon}{2}\sigma\left((z,\eta),(\tilde{z},\tilde{\eta})\right)\right)\times\\ (F_\sigma\psi)(z,\eta)(F_\sigma\varphi)(\tilde{z},\tilde{\eta})\end{array}\right\rangle_{(z,\eta)}\right\rangle_{(\tilde{z},\tilde{\eta})}=$$

$$=\left\langle\left\langle \begin{array}{l}\exp\left(2\pi\frac{2i}{\varepsilon}\sigma\left((x,p),(\tilde{z},\tilde{\eta})+(z,\eta)\right)\right)\\ \exp\left(2\pi\frac{2i}{\varepsilon}\sigma\left((\tilde{z},\tilde{\eta}),(z,\eta)\right)\right)\times\\ (F_\sigma\psi)(z,\eta)(F_\sigma\varphi)(\tilde{z},\tilde{\eta})\end{array}\right\rangle\right\rangle$$

so $\psi\#\varphi$ belongs $S\left(R^{2n}\right)$.

Let us denote K_ψ and K_φ kernels, which belong to $S\left(R^{2n}\right)$, then we have

$$\left(Dp(\psi)Dp(\varphi)\phi\right)(x)=$$

$$=\left\langle\left(K_\psi\bullet K_\varphi\right)(x,\cdot)\phi(\cdot)\right\rangle\langle\ \rangle_z=$$

$$=\left\langle\left\langle K_\psi K_\varphi(\cdot,z)\phi(z)\right\rangle\right\rangle_z=Dp(\psi\#\varphi)(x).$$

Next, using the properties of the exponential function, we have

$$\begin{aligned}
&(\psi \# \varphi)(x, p)= \\
&=\int_{R^{8n}} \left(\exp\left(2\pi i \sigma\left((x, p),(z, \eta)+(y, \varsigma)\right)\right)\right. \times \\
&\exp\left(2\pi \frac{i\varepsilon}{2} \sigma\left((z, \eta),(y, \varsigma)\right)\right) \times \\
&\exp\left(2\pi \frac{i\varepsilon}{2} \sigma\left((z, \eta),(\tilde{z}, \tilde{\eta})\right)\right) \times \\
&\exp\left(2\pi \frac{i\varepsilon}{2} \sigma\left((y, \varsigma),(\tilde{y}, \tilde{\varsigma})\right)\right) \times \\
&\psi(\tilde{z}, \tilde{\eta}) \varphi(\tilde{y}, \tilde{\varsigma})\, dz\, d\eta\, dy\, d\varsigma\, d\tilde{z}\, d\tilde{\eta}\, d\tilde{y}\, d\tilde{\varsigma}= \\
&=\int_{R^{4n}} \left(\exp\left(2\pi i \sigma\left((z, \eta),(\tilde{z}, \tilde{\eta})-(x, p)\right)\right)\right. \times \\
&\psi(\tilde{z}, \tilde{\eta}) \varphi\left((x, p)+\frac{\varepsilon}{2}(z, \eta)\right) dz\, d\eta\, d\tilde{z}\, d\tilde{\eta}
\end{aligned}$$

By changing variables $(y, \varsigma)=(x, p)+\frac{\varepsilon}{2}(z, \eta)$, we are completing the proof of the theorem.

From semigroup properties of exponential function follows: let a be a symbol of $S\left(R^{2n}\right)$ then the Weyl operator is given by

$$\hat{A}\psi(x)= \left(\frac{1}{2\pi\eta}\right)^{n}\left\langle \begin{array}{l} a\left(\frac{1}{2}(x+z), p\right) \times \\ \exp\left(\frac{i}{\eta} p \cdot(x-z)\right) \psi(z) \end{array}\right\rangle_{(z, p)},$$

the kernel of the Weyl operator A is

$$K_{\hat{A}}(x, y)=\left(\frac{1}{2\pi\eta}\right)^{n}\left\langle \begin{array}{l} \exp\left(\frac{i}{\eta} p \cdot(x-y)\right) \times \\ a\left(\frac{1}{2}(x+y), p\right) \end{array}\right\rangle_{p},$$

and the symbol is written as

$$a(x, p)= \left\langle \exp\left(-\frac{i}{\eta} p \cdot z\right) K_{\hat{A}}\left(x+\frac{1}{2} z, x-\frac{1}{2} z\right)\right\rangle_{z}.$$

These formulae are circular via the semigroup properties.

Lemma 3. ***Assume functions*** $\psi, \varphi, \phi \in S\left(R^{n}\right)$, ***then we have equality***

$$\langle\psi(\cdot), Dp(\phi) \varphi(\cdot)\rangle=\left\langle\phi(\cdot), W_{1}(\psi, \varphi)(\cdot)\right\rangle.$$

The proof follows straightforward from properties of the exponential function.

V. GENERALIZATION OF THE WEYL CALCULUS WITH *Epp* MORPHISM

Let X be a reflexive separable Banach space and $B(X, X)$ be a space of bounded operators of on X. The quantization function is a morphism $R^{n} \times R^{n} \rightarrow B(X, X)$, which maps phase space $R^{n} \times R^{n}$ to the space of bounded operators $B(X, X)$ on the reflexive separable Banach space X.

The Weyl system is defined as

$$W(x, p)=Epp\left(-l_{p} i\, \sigma((x, p),(Q, P))\right) \otimes Id(X)$$

for all points of phase space (x, p).

For any function ψ, we define the Weyl quantization by

$$Dp(\psi)=\left\langle\left(\Im_{\sigma} \psi\right)(\cdot) V(\cdot)\right\rangle$$

and the Weyl product $\psi \# \varphi$ as

$$(\psi \# \varphi)(x, p)= \left\langle\left\langle \begin{array}{l} Epp\left(l_{p} i\, \sigma((x, p),(y, q)+(z, s))\right) \times \\ \times F_{\sigma}(\psi)(y, q) F_{\sigma}(\varphi)(z, s) \\ Epp\left(l_{p} i \varepsilon\, \sigma((y, q),(z, s))\right) \end{array}\right\rangle\right\rangle.$$

The important tool of quantum physics is an asymptotic expansion of the Weyl product, which is defined as

$$\psi \# \varphi = [\psi, \varphi] - i\varepsilon\{\psi, \varphi\} + O(\varepsilon^2),$$

when the Banach space X can be equipped with its natural scalar product then the bracket $\{\psi, \varphi\}$ is Poisson brackets with the factor two; the bracket $[\psi, \varphi]$ is defined pointwise product $[\psi, \varphi](x, p) = \psi(x, p)\cdot\varphi(x, p)$.

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