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Finding on Convergence of the Flint Hills and Cookson Hills Series based on a Summation Formula of Adamchik and Srivastava involving the Riemann Zeta Function

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Abstract—This article showcases significant progress in solving two renowned problems in the calculus of series: the Flint Hills and Cookson Hills series. For almost twenty years, a long-standing question has remained unanswered in regard to their convergence. Mainly, proving the convergence of the Flint Hills series would significantly impact the redefinition of the upper bound for the irrationality measure of the number π. One of the results presented in this article is that the Flint Hills series converges to 30.3144... which leads to a redefinition of the upper bound for the irrationality measure of π, specifically $\mu(\pi) \leq 2.5$. This work proposes a transformation that solves the mystery of the Flint Hills and Cookson Hills series. It is based on a summation formula developed by mathematicians Adamchik and Srivastava. By leveraging a specialized series supported by the Riemann zeta function, this approach successfully transforms the original Flint Hills and Cookson Hills series into novel convergent versions with unique significance. The resulting sequences linked to these series are positive and bounded and satisfy convergence. Moreover, this article extends the Flint Hills series when the cosecant function has an arbitrary complex argument $n + i\beta$, with $i = \sqrt{-1}$, establishing a new series representation based on the polylogarithm $Li_3(e^{i2k})$, with $k = 1, 2, 3, \ldots$, e the Euler's number, which bears resemblance to the famous integral of the Bose-Einstein distribution as a relevant finding. This is a never-seen-before link between the Flint Hills series and polylogarithms. Furthermore, a relationship between the Apéry constant and the Flint Hills and Cookson Hills series has been established. This article presents a significant breakthrough in the calculus of series by introducing a new method based on the Riemann Zeta function and logarithmical expressions derived from the Adamchik and Srivastava summation formula. The novel approach extends the analysis of convergence criteria for series, addressing ambiguous cases characterized by abrupt jumps. Thus, the Flint Hills series converges to 30.3144... and the Cookson Hills series to 42.9949... as proved in this article.

Keywords—Apéry's constant, Cookson Hills and Flint Hills series, summation formula of Adamchik and Srivastava involving the Riemann zeta function, the upper bound for the irrationality measure of π.

I. INTRODUCTION

In 2002, Pickover, [1], introduced the Flint-Hills series as $\sum_{n=1}^{\infty} \frac{csc^2(n)}{n^3}$, where the term csc denotes the cosecant function. Despite efforts to analyze the convergence of this series, it remains an unsolved problem due to the sporadic large values and unexpected jumps of $csc^2(n)$ in the plots of the partial sums up to $n = 10^4$, [2]. Some statistical results, such as, [3], suggest that the Flint Hills series tends towards the numerical value of $30.3144\ldots$ However, there is no known method to solve that mystery which makes the Flint-Hills series an extremely difficult problem in mathematical analysis and calculus. Moreover, the most interesting fact about this series is that the behavior of its partial sums is closely connected to the rational approximations to π. For example, [4], proved that convergence of the Flint Hills series would imply a new upper bound of $\mu(\pi) \leq 2.5$ for the irrationality measure of π. In fact, [5], established the current upper bound for the irrationality measure of π, i.e., $\mu(\pi) \leq 7.6063\ldots$. Furthermore, as described the irrationality measure μ(x) of a positive real number x is defined as the infimum of such m that the inequality $0 < \left|x - \frac{p}{q}\right| < \frac{1}{q^m}$ holds only for a finite number of co-prime positive integers p and q. If no such m exists, then $\mu(x) = +\infty$ and x is called Liouville number, as a result, the larger is μ(x), the better x is approximated by rational numbers. Thus, if $\mu(x) = 1$ means that x is a rational number and for the other cases $\mu(x) = 2$ and $\mu(x) \geq 2$ that x is an irrational algebraic number and x is a transcendental number respectively. Therefore, there exists a strong connection between number theory, specifically the irrationality measure of π, and the solution of the Flint Hills series problem due to its trigonometric nature of it, as evident in the existing literature. This paper aims to propose a novel representation, which could establish a new convergence criterion for special cases of these series in the future, for computing the convergent values of the Flint Hills and Cookson Hills series, and other known general cases of these series that have not been understood yet. This new representation utilizes a formula that translates the observed ambiguity in convergence for both renowned series into a stable convergence pattern linked to positive and bounded sequences that avoid the original jumps computed by the partial sums of these series, ultimately yielding the well-established and statistically proven values of 30.3144... for the Flint Hills series and 42.9949... for the Cookson Hills series as expected.

The mathematical analysis is also related to how to address the challenges in solving the Cookson Hills series $\sum_{n=1}^{\infty} \frac{sec^2(n)}{n^3}$, [6], with sec the secant function, because this series is closely linked to the Flint Hills series. Essentially, solving the Flint Hills series also addresses the convergence status of the Cookson Hills series, which is often overlooked or not mentioned explicitly. Pickover, in 2002, [1], introduced this enigmatic series with the same difficulty level of analysis because it is not known if the Cookson Hills series converges, since $sec^2(n)$ can have also sporadic large values as known. Some other examples of interesting series can be also $\sum_{n=1}^{\infty} \frac{tan^2(n)}{n^3}$ and $\sum_{n=1}^{\infty} \frac{cot^2(n)}{n^3}$, which will be also analyzed in this article as findings. The importance of these unsolved problems in the calculus of series is often underestimated, as they are rarely mentioned in the literature on series and sequences. Moreover, this article establishes an intriguing connection between this kind of series and certain concepts in physics, owing to the prevalence of polylogarithms in an extended version of the original Flint Hills series, which will be discussed later.

To obtain a useful tool for addressing the convergence issue of these series, this work introduces an important relationship discovered by [7]. This relationship is a summation formula that the authors listed as *(4.8)* in their section *4, Series Involving Polylogarithmic Functions,* i.e., $\sum_{m=1}^{\infty} \frac{t^m}{m^2} \zeta(2m) = log(\pi\sqrt{t}\, csc(\pi\sqrt{t}\,))$. The term csc corresponds precisely to the cosecant function used in the Flint Hills series; ζ is the Riemann Zeta function. Thus, it would be useful to approach the numerator $csc^2(n)$ in the Flint Hills series by a trick that has never been attempted before.

Adamchik and Srivastava derived their formula after manipulating the equation *(4.5)* of their article, i.e., the expression $L_{(n)} := \sum_{n=2}^{\infty} (-1)^n Li_2\left(\frac{4}{n^2}\right)$, and replacing the polylogarithmic function with its series representation given by their equation *(1.12)*, i.e., $Li_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s}$, with $s = 2$, then, they changed the order of summation, and evaluated the inner sum, a procedure that led to getting a fascinating formula listed as *(4.6)*, $L_{(n)} = 2\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k^2} - \sum_{k=1}^{\infty} \{\zeta(2k) - 1\} \frac{4^k}{k^2}$.

The equation *(4.6)* was combined with another relationship listed as *(4.7)* by these authors, resulting in the following expression $\sum_{k=1}^{\infty} \frac{t^k}{k^2} \zeta(2k) = \int_0^1 \frac{dt}{t} \sum_{k=1}^{\infty} \frac{t^k}{k} \zeta(2k)$. After a detailed evaluation of the inner sum, they derived the formula listed as *(4.8)* which serves as the main tool in this paper to validate the convergence of the Flint Hills and Cookson Hills series. The methods for evaluating the inner sum have been explained in detail in their article. Thanks to them, the known relationship (4.8), $\sum_{m=1}^{\infty} \frac{t^m}{m^2} \zeta(2m) = log(\pi\sqrt{t}\, csc(\pi\sqrt{t}\,))$, has a formal derivation and even is validated when using it for the convergence of the Flint Hills and Cookson Hills series.

In terms of future research directions in the field of these particular unsolved series and the potential application of the relationship (4.8) within the framework of a future formal criterion, this article elucidates why solving the Flint Hills and Cookson Hills series problems would provide the fundamental groundwork for addressing general cases such as the generalized Flint Hills series $\sum_{n=1}^{\infty} \frac{csc^v(n)}{n^u}$ or the generalized Cookson Hills series $\sum_{n=1}^{\infty} \frac{sec^v(n)}{n^u}$ with various integers or even real values for u and v. This paper presents a significant finding regarding the existence of a fundamental formula, as introduced by Adamchik and Srivastava in their paper, namely equation (4.8), which allows for the convenient adjustment of the cosecant function to transform the inherent complexity of the original Flint Hills series problem and subsequently, the Cookson Hills series problem, into a manageable set of series that adhere to criteria of boundedness and non-diverging positive terms. This is the significant contribution of the present article. Subsequently, researchers can comprehend the distinctive aspects that set this work apart from previously published studies on the subject. This paper presents the sole known method that establishes the convergence of both the Flint Hills and Cookson Hills series unequivocally. The numerical evidence obtained through these formulas as findings align with the anticipated values of convergence, supported by statistical analysis.

II. Proof Of Convergence of the Flint Hills Series

As previously mentioned, Adamchik and Srivastava introduced their formula, denoted as (4.8), [1],

$$\sum_{m=1}^{\infty} \frac{t^m}{m^2} \zeta(2m) = log(\pi\sqrt{t}\, csc(\pi\sqrt{t}\,)), \quad (1)$$

the hitherto unnoticed use of this relationship within the context of the Flint Hills and Cookson Hills series lies in the fact that the variable t can assume several numerical values whenever $t < 1$ to avoid imprecise domains of convergence that do not correspond to the validity of (1). Therefore, I consider the real values of t as follows

$$t = \frac{1}{\pi^2} e^{-2csc^2(n)}, \tag{2}$$

where $e = 2.71828 \ldots$ is the Euler's number, $\pi = 3.1415 \ldots$ is the mathematical constant pi, $csc(n) = \frac{1}{sin(n)}$ represents the cosecant function based on sine, and $n = 1, 2, 3, \ldots$ denotes the set of non-negative integers from the series expansion of the Flint-Hills series. In equation (2), there is no specific value of n that can cause divergence in $csc^2(n)$ and exceed t to a value greater than 1. This is because the exponential term $e^{-2csc^2(n)}$ never exhibits such behavior. Thus, I establish the value of t in this manner. Furthermore, $csc^2(n)$ plays a crucial role in the representation of the Flint Hills series within this approach. I substitute equation (2) into equation (1) by following these algebraic steps

$$\sum_{m=1}^{\infty} \frac{e^{-2m\, csc^2(n)}}{m^2 \pi^{2m}} \zeta(2m) = log(\pi \sqrt{\frac{1}{\pi^2} e^{-2csc^2(n)}}\, csc(\pi \sqrt{\frac{1}{\pi^2} e^{-2csc^2(n)}})),$$

$$\sum_{m=1}^{\infty} \frac{e^{-2m\, csc^2(n)}}{m^2 \pi^{2m}} \zeta(2m) = log(e^{-csc^2(n)}\, csc(e^{-csc^2(n)})),$$

$$\sum_{m=1}^{\infty} \frac{e^{-2m\, csc^2(n)}}{(m\, \pi^m)^2} \zeta(2m) = log(1) - log e^{csc^2(n)} + log(csc(e^{-csc^2(n)})),$$

$$\sum_{m=1}^{\infty} \frac{e^{-2m\, csc^2(n)}}{(m\, \pi^m)^2} \zeta(2m) = -csc^2(n) + log(csc(e^{-csc^2(n)})),$$

$$csc^2(n) = log\ [csc(e^{-csc^2(n)})] - \sum_{m=1}^{\infty} \frac{e^{-2m\, csc^2(n)}}{(m\, \pi^m)^2} \zeta(2m).$$

It is evident that $csc^2(n)$ can be divided by n^3 in the next step, as demonstrated below

$$\frac{csc^2(n)}{n^3} = \frac{log\ [csc(e^{-csc^2(n)})]}{n^3} - \frac{1}{n^3} \sum_{m=1}^{\infty} \frac{e^{-2m\, csc^2(n)}}{(m\, \pi^m)^2} \zeta(2m), \tag{3}$$

with $\frac{log\,[csc(e^{-csc^2(n)})]}{n^3} = n^{-3} log\,[csc(e^{-csc^2(n)})]$ or also $\frac{log\ [csc(e^{-csc^2(n)})]}{n^3} = log\ \{csc(e^{-csc^2(n)})\}^{n^{-3}}$.

Thus, I get

$$\frac{csc^2(n)}{n^3} = log\{csc(e^{-csc^2(n)})\}^{n^{-3}} - \frac{1}{n^3} \sum_{m=1}^{\infty} \frac{e^{-2m\, csc^2(n)}}{(m\, \pi^m)^2} \zeta(2m). \tag{4}$$

Based on (3) and (4), I represent the Flint Hills series through the following equivalent versions, (5) and (6),

$$\sum_{n=1}^{\infty} \frac{csc^2(n)}{n^3} = \sum_{n=1}^{\infty} \frac{log\ csc(e^{-csc^2(n)})}{n^3} - \sum_{n=1}^{\infty} \frac{1}{n^3} \sum_{m=1}^{\infty} \frac{e^{-2m\, csc^2(n)}}{(m\, \pi^m)^2} \zeta(2m), \tag{5}$$

$$\sum_{n=1}^{\infty} \frac{csc^2(n)}{n^3} = \sum_{n=1}^{\infty} log\ \{csc(e^{-csc^2(n)})\}^{n^{-3}} - \sum_{n=1}^{\infty} \frac{1}{n^3} \sum_{m=1}^{\infty} \frac{e^{-2m\, csc^2(n)}}{(m\, \pi^m)^2} \zeta(2m), \tag{6}$$

Definition 1. The Flint-Hills-López Series Representation

The Flint-Hills-López series representation for $\sum_{n=1}^{\infty} \frac{csc^2(n)}{n^3}$ is given by subtracting two convergent series S_{L_1} and S_{L_2} derived from the analysis of the Adamchik-Srivastava Summation Formula, $\sum_{m=1}^{\infty} \frac{t^m}{m^2} \zeta(2m) = log(\pi\sqrt{t}\, csc(\pi\sqrt{t}))$, when $t = \frac{1}{\pi^2} e^{-2csc^2(n)}$, such that S_{L_1} converges to $\tau = 30.326256 \ldots$ and S_{L_2} converges to $\sigma = 0.0118355169 \ldots$.

$$S_{L_1} = \sum_{n=1}^{\infty} log\ \{csc(e^{-csc^2(n)})\}^{n^{-3}} = \tau = 30.326256 \ldots,$$

$$S_{L_2} = \sum_{n=1}^{\infty} \frac{1}{n^3} \sum_{m=1}^{\infty} \frac{e^{-2m\, csc^2(n)}}{(m\, \pi^m)^2} \zeta(2m) = \sigma = 0.01183551 \ldots,$$

$$\sum_{n=1}^{\infty} \frac{csc^2(n)}{n^3} = S_{L_1} - S_{L_2} = \tau - \sigma = 30.3144204831 \ldots.$$

The result in *Definition 1* meets a very close value of convergence statistically proved by [3], and seen in various plots of series expansion for the Flint Hills series, e.g., in the document of Wolfram MathWorld, [2], where that series seemed to converge to a very similar number $30.314 \ldots$, which is the result of the partial sums calculated for the Flint Hills series, e.g., up to $n = 10^4$. Moreover, convergence on S_{L_1} is established definitely because there is a positive function $\alpha(x) = \frac{log\ csc(e^{-csc^2(x)})}{x^3} = log\ \{csc(e^{-csc^2(x)})\}^{x^{-3}}$ raised on $x \in [1, \infty)$ such that the area $A = \int_1^{\infty} \alpha(x) dx$ under the curve $\alpha(x)$ is positive and finite which respects the criterion of boundedness applied to the series S_{L_1}. Fig.1 shows that the area under the curve naturally vanishes slowly in the infinite, despite the fact that there are some *sharp spikes* of discontinuity related to $\alpha(x)$. Fig. 2 illustrates the gradual decrease of $\alpha(x)$ along the x-axis, indicating its slow vanishing as x increases. The value of $\alpha(x)$ diminishes without approaching infinity.

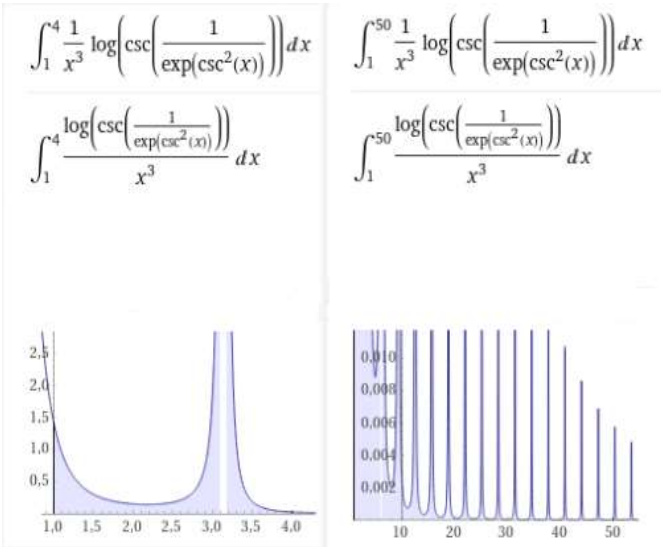

Figure 1. Plot of the area under the curve α(x) vanishing slowly along the axis x. α(x) contains 'spikes' of infinite +∞ at the first non-integer values of x. However, the respective sequence α(n) is not divergent on the integers x=n, n=1,2,3...

Being $\alpha(x_{0\,discont}) = \infty$. Thus, these points $x_{0\,discont}$ belong to the real function $\alpha(x)$ and do not appear in the sequence $\alpha(n) = log\ \{csc(e^{-csc^2(n)})\}^{n^{-3}}$, which is a verifiable fact if tested in n. Moreover, Fig 3 let us depict that phenomenon. Therefore, the bars or samples $\alpha(n) = \{\alpha_{(1),}\alpha_{(2),}\alpha_{(3),} \dots\}$ are always positive, thanks to the natural logarithm and its argument elevated at n^{-3}, and also each $\alpha(n)$ is bounded which means that $\alpha(n)$ is reducing its size slowly when n grows and is vanishing in the infinite as the expected behavior for a convergent series that represents the main contribution of the Flint Hills series. It is clear, based on the numerical data and plots, that there are zones of the total shadow area under the curve $\alpha(x)$, in Fig.1, that are close to certain *spikes of infinite* value and to other lower sizes but it does not mean that the area between such points was unbounded or infinite. The sequence $\alpha(n) = log\ \{csc(e^{-csc^2(n)})\}^{n^{-3}}$ is established in Lemma 1.

Lemma 1. The sequence $\{\alpha(n) = log\ \{csc(e^{-csc^2(n)})\}^{n^{-3}}\}$ associated to the natural numbers, N, i.e., $n = 1, 2, 3, \dots$, achieves the definition of bounded sequences given by:

There exists a real number $M = 1.8597 \dots$ such that $|\alpha(n)| \leq M$ for all natural numbers N, $|\ n = 1, 2, 3, \dots$. Thus, the sequence is bounded as $0 < |\alpha(n)| \leq 1.8597 \dots$ being $\alpha(n)$ always positive.

Proof. The plot of every sample $\alpha(n) = \{\alpha_{(1),}\alpha_{(2),}\alpha_{(3),} \dots\}$

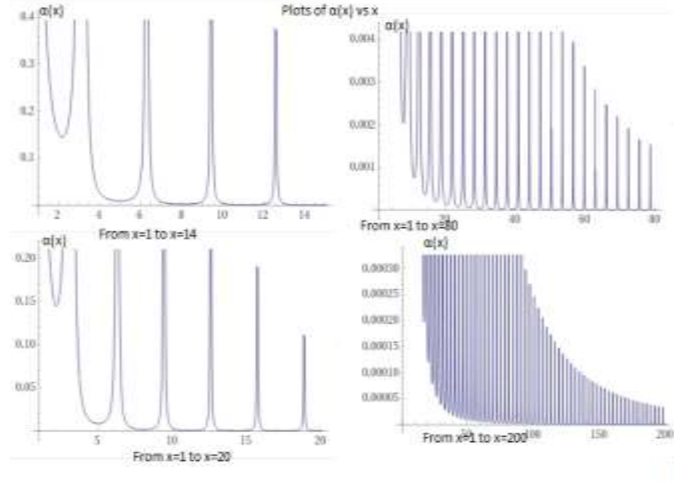

Figure 2 Plot of the curve α(x) vanishing slowly along the axis x. The spikes are decreasing (non-infinite) as x grows.

, with $\alpha(n) = log\ \{csc(e^{-csc^2(n)})\}^{n^{-3}}$, shows a slow-damped behavior regarding $\alpha(n)$. The highest visible sample of the whole set is $\alpha_{(3)} = M = 1.8597 \dots$ while the rest of the samples are less than $\alpha_{(3)} = M$. However, when $n \to \infty$ the samples are vanishing in the infinite if their height is computed and analyzed, so the natural tendency of the sequence $\alpha(n)$ is being bound, there will not be divergence as n increases! However, the process of achieving the number $\tau = 30.326256 \dots$ associated to S_{L_1} demands calculating several initial samples distributed over the first thousands of integers from $n = 1$ to $n = 2200$ approximately. The algorithms can easily compute the first 2200 samples or even more, like 10000 samples, in less than 1 minute using efficient software like Wolfram Alpha, [8], and prove that the main values of the sequence $\alpha(n)$ used in the summation to compute S_{L_1}, i.e., the set of blue bars in Fig.3 in a plot of Matlab, [9], are enough to calculate $\tau = 30.326256 \dots$ which is the main number that supports the total sum to compute the Flint Hills series. Thus, samples for higher $n > 2200$ will contribute by adding more precise decimals to $\tau = 30.326256 \dots$ but the integer part 30 will remain as the main value in that series.

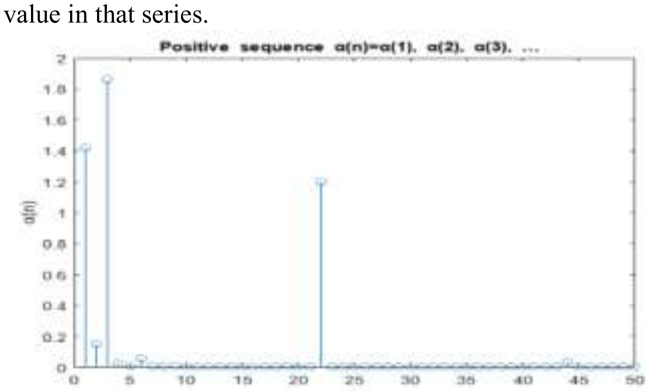

Figure 3. The first 50 samples of α(n) in Matlab. The sequence is decreasing but there are significantly smaller samples further than n=50 whose contribution is relevant to this analysis. $\alpha(x)$ is stable and perfectly computable within 2200 samples.

I share the simple line of code in WolframAlpha, [8], written in natural language, that everybody can use to verify convergence for $S_{L_1} = \tau = 30.326256\ldots$ by the code

Sum[1/(n^3)log(csc(1/exp(csc(n)^2))), {n,1,4200}], or also

*sum (1/(n^3)*log(csc(1/exp(csc(n)^2)))) from n=1 to 4200*,

of course, the number of samples can be adjusted, e.g., just replacing 4200 with higher numbers like 9000 or 10000 to improve precision. Moreover, other kinds of software packages can prove the same computed results if their precision is well established. Fig. 4 shows a typical routine online in WolframAlpha, [8], that let us accomplish the value of $S_{L_1} = \tau = 30.326256\ldots$

Figure 4. Computation of the main series S_{L_1} which converges to $\tau = 30.326256\ldots$

Lemma 2. The series S_{L_2} converges notoriously quickly thanks to the exponential $e^{-2m\,csc^2(n)}$ which decays rapidly and also because of the damping effect of $(m\,\pi^m)^2$ in the denominator, i.e., $\frac{e^{-2m\,csc^2(n)}}{(m\,\pi^m)^2} = \frac{1}{(m\,\pi^m)^2\,e^{2m\,csc^2(n)}}$ that decays quickly without infinities. Therefore, convergence for $S_{L_2} = \sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{m=1}^{\infty}\frac{e^{-2m\,csc^2(n)}}{(m\,\pi^m)^2}\,\zeta(2m)$ is given by $\sigma = 0.0118355169\ldots$. $S_{L_2} = \sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{m=1}^{\infty}\frac{e^{-2m\,csc^2(n)}}{(m\,\pi^m)^2}\,\zeta(2m)$ is called the Minor Zeta Flint-Hills-Lopez series.

Proof. The Lemma 2 is proved numerically by computing the first integers $n = 1, 2, 3, 4$ when m is great enough, e.g., $m = 8000$, to be considered as $m \to \infty$ within the inner large partial sum given by $\sum_{m=1}^{\infty}\frac{e^{-2m\,csc^2(n)}}{(m\,\pi^m)^2}\,\zeta(2m)$ and after exposing the fact that the expansion of the sample for $n > 4$ is definitely very close to zero. The code in Wolfram Alpha is provided for each case of $n = 1, 2, 3, 4$ as follows

For $n = 1 \;\to\; \frac{1}{n^3}\sum_{m=1}^{\infty}\frac{e^{-2m\,csc^2(n)}}{(m\,\pi^m)^2}\,\zeta(2m) \cong 0.00989887$, i.e., $\sum_{m=1}^{8000}\frac{1}{(1^3)\,e^{2m\,csc^2(1)}(m\,\pi^m)^2}\,\zeta(2m) \cong 0.00989887$, with code

*Sum[Divide[ζ\(40)2m\(41)Power[1,-3],Square[\(40)m*Power[π,m]\(41)]exp\(40)2*m*Square[\(40)csc\(40)1\(41)\(41)]\(41)],{m,1,8000}]*

For $n = 2$; $\sum_{m=1}^{8000}\frac{1}{(2^3)\,e^{2m\,csc^2(2)}(m\,\pi^m)^2}\,\zeta(2m) \cong 0.00185733$, with code

*Sum[Divide[ζ\(40)2m\(41)Power[2,-3],Square[\(40)m*Power[π,m]\(41)]exp\(40)2*m*Square[\(40)csc\(40)2\(41)\(41)]\(41)],{m,1,8000}]*

For $n = 3$; $\sum_{m=1}^{8000}\frac{1}{(3^3)\,e^{2m\,csc^2(3)}(m\,\pi^m)^2}\,\zeta(2m) \cong 0$, with code

*Sum[Divide[ζ\(40)2m\(41)Power[3,-3],Square[\(40)m*Power[π,m]\(41)]exp\(40)2*m*Square[\(40)csc\(40)3\(41)\(41)]\(41)],{m,1,8000}]*

For $n = 4$; $\sum_{m=1}^{8000}\frac{1}{(4^3)\,e^{2m\,csc^2(4)}(m\,\pi^m)^2}\,\zeta(2m) \cong 0.0000793169$, with code

*Sum[Divide[ζ\(40)2m\(41)Power[4,-3],Square[\(40)m*Power[π,m]\(41)]exp\(40)2*m*Square[\(40)csc\(40)4\(41)\(41)]\(41)],{m,1,8000}]*

WolframAlpha starts calculating successive values practically equal to zero from n=5 on as seen in Fig.5. The code for the next integers n=5 and n=6 is available below

*Sum[Divide[ζ\(40)2m\(41)Power[5,-3],Square[\(40)m*Power[π,m]\(41)]exp\(40)2*m*Square[\(40)csc\(40)5\(41)\(41)]\(41)],{m,1,8000}]*

*Sum[Divide[ζ\(40)2m\(41)Power[6,-3],Square[\(40)m*Power[π,m]\(41)]exp\(40)2*m*Square[\(40)csc\(40)6\(41)\(41)]\(41)],{m,1,8000}]*

Figure 5. Computation of $\frac{1}{n^3}\sum_{m=1}^{\infty}\frac{e^{-2m\,csc^2(n)}}{(m\,\pi^m)^2}\,\zeta(2m) \approx 0$ for $n = 5$ and $n = 6$ via Wolfram Alpha.

The sum for S_{L_2} is $\sigma \approx 0.00989887 + 0.00185733 + 0 + 0.0000793169 + 0 + 0 \ldots \approx 0.0118355169$. Therefore, S_{L_2} is smaller than S_{L_1}, yet it remains a significant value in achieving convergence for the Flint Hills series.

III. The Apéry Constant, $\zeta(3)$, and its Relationship with the Flint Hill Series and The Series $\sum_{n\geq1}\frac{cot^2(n)}{n^3}$

I derive a second new representation for the Flint Hills series based on the well-known fundamental trigonometric identity, [10], as below

$$csc^2(\theta) = 1 + cot^2(\theta), \quad (7)$$

which is easily inferred from the Pythagorean trigonometric identity $sin^2(\theta) + cos^2(\theta) = 1$. Clearly, (7) implies that every real value of θ is possible, even an integer $\theta = n$, with $n = 1, 2, 3, \ldots$. Thus, I notice that the following version of (7)

$$csc^2(n) - cot^2(n) = 1, \quad (8)$$

can be modified by multiplying both sides of (8) by $\frac{1}{n^3}$ as follows

$$\frac{csc^2(n)}{n^3} - \frac{cot^2(n)}{n^3} = \frac{1}{n^3}, \quad (9)$$

obviously with $n \neq 0$ within this context. Then, I let $\{\frac{csc^2(n)}{n^3} : n \geq 1\}$ be the sequence related to the Flint Hills series $\sum_{n\geq1}\frac{csc^2(n)}{n^3}$ and $\{\frac{cot^2(n)}{n^3} : n \geq 1\}$ a new sequence derived from the second term $\frac{cot^2(n)}{n^3}$ seen on the left side of (9). Therefore, I establish a relevant convergent series given by

$$\sum_{n\geq1}\frac{cot^2(n)}{n^3}, \quad (10)$$

and also the notorious Apéry's constant, $\zeta(3)$, given by the definition of the sequence $\{\frac{1}{n^3} : n \geq 1\}$

$$\sum_{n\geq1}\frac{1}{n^3} = \zeta(3). \quad (11)$$

I combine (10) and (11), for all integers $n \geq 1$, to yield a valid relationship that involves the Flint Hills series and the proposed series (10) and (11) within a never seen previous scenario of convergence for this unsolved problem in mathematical analysis.

Definition 2. The Pythagorean-Flint-Hills series representation is defined by

$$\sum_{n\geq1}\frac{csc^2(n)}{n^3} - \sum_{n\geq1}\frac{cot^2(n)}{n^3} = \zeta(3), \quad (12)$$

where $\sum_{n\geq1}\frac{csc^2(n)}{n^3}$ is the Flint Hills series, $\sum_{n\geq1}\frac{cot^2(n)}{n^3}$ converges to a finite value φ, and $\zeta(3)$ is called the Apéry constant, the value of the Riemann zeta function at 3 or $\zeta(3) =$ 1.2020569031595942853997381615114499907...., which has multiple mathematical representations.

Lemma 3. The series $\sum_{n\geq1}\frac{cot^2(n)}{n^3} = \varphi = \tau - \sigma - \zeta(3) \approx 30.326256 - 0.0118355169 - 1.2020569 \approx 29.1123635831.$

Proof. A direct proof is involving *Definition 1, Lemma 1,* and *Lemma 2* which leads to calculating immediately such value of convergence $\varphi \approx 29.1123635831$. However, a second approach is to find a new convergent series representation for $\sum_{n\geq1}\frac{cot^2(n)}{n^3}$ using the relationship given by (1) and establishing $t = \frac{1}{\pi^2}e^{-2cot^2(n)}$ as follows

Let $t = \frac{1}{\pi^2}e^{-2cot^2(n)}$ be replaced in the expression

$$\sum_{m=1}^{\infty}\frac{t^m}{m^2}\zeta(2m) = log(\pi\sqrt{t}\, csc(\pi\sqrt{t}\,))$$

$$\sum_{m=1}^{\infty}\frac{e^{-2m\,cot^2(n)}}{\pi^{2m}m^2}\zeta(2m) = log(\pi\sqrt{\frac{1}{\pi^2}e^{-2cot^2(n)}}\, csc(\pi\sqrt{\frac{1}{\pi^2}e^{-2cot^2(n)}})),$$

then rules of algebra lead to get

$$\sum_{m=1}^{\infty}\frac{e^{-2m\,cot^2(n)}}{\pi^{2m}m^2}\zeta(2m) = -cot^2(n) + log\, csc(e^{-cot^2(n)}),$$

dividing by n^3 in both sides as shown below

$$\frac{1}{n^3}\sum_{m=1}^{\infty}\frac{e^{-2m\,cot^2(n)}}{\pi^{2m}m^2}\zeta(2m) = -\frac{1}{n^3}cot^2(n) + \frac{1}{n^3}log\, csc(e^{-cot^2(n)}),$$

and just establishing the sum

$$\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{m=1}^{\infty}\frac{e^{-2m\,cot^2(n)}}{\pi^{2m}m^2}\zeta(2m) = -\sum_{n=1}^{\infty}\frac{1}{n^3}cot^2(n) + \sum_{n=1}^{\infty}\frac{1}{n^3}log\, csc(e^{-cot^2(n)}),$$

which leads to represent

$$\sum_{n=1}^{\infty}\frac{cot^2(n)}{n^3} = \sum_{n=1}^{\infty}\frac{1}{n^3}log\, csc(e^{-cot^2(n)}) - \sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{m=1}^{\infty}\frac{e^{-2m\,cot^2(n)}}{\pi^{2m}m^2}\zeta(2m), \quad (13)$$

where $\sum_{n=1}^{\infty}\frac{1}{n^3}log\, csc(e^{-cot^2(n)}) \approx 29.20304786$ based on Wolfram Alpha by the code indicated below
*sum (1/(n^3)*log(csc(1/exp(cot(n)^2)))) from n=1 to 9000*

, and the expression

$\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{m=1}^{\infty}\frac{e^{-2m\,cot^2(n)}}{\pi^{2m}m^2}\zeta(2m) \approx 0.073615 + 0.0138581 + 0 + 0.000587 + 0 \ldots \approx 0.0880601$. Therefore, as a result

$\sum_{n=1}^{\infty} \frac{cot^2(n)}{n^3} \approx 29.20304786 - 0.0880601 \approx 29.11498776$
which is very close to $\varphi \approx 29.1123635831$

Definition 3. The López Flint-Hills representation involving the Apéry constant is given by the relations

$$\zeta(3) = \sum_{n\geq 1} \frac{csc^2(n)}{n^3} - \sum_{n\geq 1} \frac{cot^2(n)}{n^3},$$

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{log\ csc\left(e^{-csc^2(n)}\right)}{n^3} - \sum_{n=1}^{\infty} \frac{1}{n^3} \sum_{m=1}^{\infty} \frac{e^{-2m\, csc^2(n)}}{(m\, \pi^m)^2}\, \zeta(2m) - \sum_{n=1}^{\infty} \frac{1}{n^3} log\ csc(e^{-cot^2(n)}) + \sum_{n=1}^{\infty} \frac{1}{n^3} \sum_{m=1}^{\infty} \frac{e^{-2m\, cot^2(n)}}{(m\, \pi^m)^2} \zeta(2m)$$

which exposes a fascinating result because of the difference between the Flint Hills series and $\sum_{n\geq 1} \frac{cot^2(n)}{n^3}$ is, in fact, the Apéry constant, and $\sum_{n\geq 1} \frac{csc^2(n)}{n^3}$ and $\sum_{n\geq 1} \frac{cot^2(n)}{n^3}$ are two convergent series, remarkable findings that had not been discovered until the present paper. Moreover, I highlight the fact that even if both series had diverged, the expression for $\zeta(3)$ would have respected also nature for raising the Apéry constant. Therefore, the Flint Hills series is written as

$$\sum_{n\geq 1} \frac{csc^2(n)}{n^3} = \zeta(3) + \sum_{n=1}^{\infty} \frac{1}{n^3} log\ csc(e^{-cot^2(n)}) - \sum_{n=1}^{\infty} \frac{1}{n^3} \sum_{m=1}^{\infty} \frac{e^{-2m\, cot^2(n)}}{(m\, \pi^m)^2} \zeta(2m) \approx 30.3144204831. \quad (14)$$

IV. THE NEW UPPER BOUND FOR THE IRRATIONALITY MEASURE OF π IS GIVEN BY $\mu(\pi) \leq \frac{5}{2}$

In [11], the author revised the matter of whether the Flint Hills series converges or not and pointed out the article of [4], who connected this question to the irrationality measure of π, that $\mu(\pi) > \frac{5}{2}$ would imply divergence of the Flint-Hills series and $\mu(\pi) < \frac{5}{2}$ convergence. Moreover, [11], showed that convergence would imply that $\mu(\pi) \leq \frac{5}{2}$, i.e., $\mu(\pi) \leq 2.5$. Therefore, I have proved that the irrationality measure of π is less than or equal to 2.5, i.e., $\mu(\pi) \leq 2.5$, based on the finding of the relationships (5), (6) or *Definition 1, Lemma 1 and Lemma 2*. Moreover, in the case where $u = 3$ and $v = 2$ for the Flint-Hills series, i.e., in $\sum_{n=1}^{\infty} \frac{csc^v(n)}{n^u} = \sum_{n=1}^{\infty} \frac{csc^2(n)}{n^3}$, Meiburg concluded that this sum converges whenever $\mu(\pi) < \frac{3+\sqrt{3}}{2} \cong 2.366$. Therefore, my finding of a new convergent representation for the Flint Hills series lets conclude the veracity of Corollary 4 of [4], textually

Corollary 4. If the Flint Hills series, $\sum_{n=1}^{\infty} \frac{csc^2(n)}{n^3}$, converges, then $\mu(\pi) \leq \frac{5}{2}$.

Proof. Convergence of $\sum_{n=1}^{\infty} \frac{csc^2(n)}{n^3}$ implies that $\lim_{N\to\infty} \frac{1}{n^3 sin^2(n)} = 0$ and thus by *Corollary 3*, $\mu(\pi) \leq \frac{5}{2}$.

Corollary 3. For positive real numbers u and v,

1. If the sequence $\frac{1}{n^u|sin^v(n)|}$ converges, then $\mu(\pi) \leq 1 + \frac{u}{v}$;
2. If the sequence $\frac{1}{n^u|sin^v(n)|}$ diverges, then $\mu(\pi) \geq 1 + \frac{u}{v}$;

V. MODIFYING THE LAST KNOWN UPPER BOUND FOR THE IRRATIONALITY MEASURE OF π: $\mu(\pi) < 7.6063$ ... NEW IMPLICATIONS DERIVED FROM $\mu(\pi) \leq 2.5$

The best currently known upper bound $\mu(\pi) < 7.6063$ ... was obtained in 2008 by [5], now, thanks to my finding in this article the new upper bound can be established as $\mu(\pi) \leq 2.5$. In the references, by Theorem 2, [5], Salikhov's bound implies that the sequence $\frac{1}{n^u|sin(n)|^v}$ converges to zero as soon as $1 + \frac{u}{v} > 7.6063$ ... Therefore, the new perspectives based on my finding are focused on solving particular cases for the pairs $(u, v) = (7, 1), (14, 2), (20, 3)$, and for other higher cases, and to prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^u|sin(n)|^v}$ converges for $(u, v) = (8, 1), (15, 2), (21, 3)$, etc. In the discussion section of this paper, I provide perspectives and the direction of this work for the unsolved cases related to the pairs (u, v). Positive and bounded sequences derived from the Adamchik-Srivastava formula could be combined with my proposal to define a criterion for all the pairs (u, v) and establish an important branch in the calculus of series. Another important implication related to the recent finding for the Flint Hills series and $\mu(\pi) \leq 2.5$ is provided by [12], in Researchgate, who exposes in an interesting discussion that *"... as π is just a ratio of circumference to the square of the radius, then it would mean the continuity or integrability of the 1-D (one dimensional) surface area of the circle is actually a lot smaller than we think for a unit circle."* This implication has been proved based on the current finding of this article and considered in the discussion section as well.

VI. GENERALIZATION OF THE FLINT HILLS SERIES WITHIN THE COMPLEX DOMAIN: THE COMPLEX FLINT-HILLS-LÓPEZ SERIES

I establish the Complex Flint-Hills-López Series, $Ł_z(\beta)$, as follows

$$Ł_z(\beta) := \sum_{n=1}^{\infty} \frac{csc^2(n + i\beta)}{n^3}, \quad (15)$$

where β is any given real number and $i = \sqrt{-1}$. Moreover, the original Flint Hills series is evaluated when $\beta = 0$, i.e., $Ł_z(0) = \tau - \sigma = 30.3144204831\ldots$ as proved before.

In this paper, I derive a particular expression for $Ł_z(\beta)$, $\beta \neq 0$, by using the q-series, [13], for the cotangent function $cot(z)$, with $q = e^{iz}$, defined below

$$cot(z) = -i - 2i\sum_{k=1}^{\infty} q^{2k} = -i - 2i\sum_{k=1}^{\infty} e^{2ikz}, \quad (16)$$

then, I calculate the first derivative of $cot(z)$ and also for its q-series with respect to z

$$\frac{d(cot(z))}{dz} = -csc^2(z) = -i\frac{d(1)}{dz} - 2i\frac{d(\sum_{k=1}^{\infty} e^{2ikz})}{dz} = -2i\sum_{k=1}^{\infty}\frac{d}{dz}(e^{2ikz}) = 4\sum_{k=1}^{\infty} ke^{2ikz} = \frac{4\,e^{2iz}}{(1-e^{2iz})^2}, \quad (17)$$

in the mathematical references is already known the sum $\sum_{k=1}^{\infty} ke^{2ikz} = \frac{4\,e^{2iz}}{(1-e^{2iz})^2}$; a simple computation on WolframAlpha, helps to verify it. Thus, I use (17) to represent $\frac{csc^2(n+i\beta)}{n^3}$ In the case of the Flint-Hills-López Series $Ł_z(\beta)$ when $z = n + i\beta$

$$\frac{csc^2(n+i\beta)}{n^3} = -4\frac{e^{2i(n+i\beta)}}{n^3(1-e^{2i(n+i\beta)})^2} = -4\frac{e^{2in}\,e^{-2\beta}}{n^3(1-e^{2in}\,e^{-2\beta})^2} = -4\frac{e^{2in}\,e^{2\beta}}{n^3(e^{2\beta}-e^{2in})^2} = -4\frac{e^{2in}\,e^{2\beta}}{n^3(e^{2in}-e^{2\beta})^2}, \quad (18)$$

$$Ł_z(\beta) = \sum_{n=1}^{\infty}\frac{csc^2(n+i\beta)}{n^3} = -4\sum_{n=1}^{\infty}\frac{e^{2in}\,e^{2\beta}}{n^3(e^{2in}-e^{2\beta})^2} = -4e^{2\beta}\sum_{n=1}^{\infty}\frac{e^{2in}}{n^3(e^{2in}-e^{2\beta})^2}. \quad (19)$$

No singularities are originating from the denominator of the addends of (19) because $n^3 \neq 0$ and $e^{2in} - e^{2\beta} \neq 0$ for every n > 0. Moreover, a trick to evaluate (19) is just to observe that if $q = e^{2in}$, then $\frac{q}{(q-e^{2\beta})^2} = \frac{q}{e^{4\beta}} + \frac{2q^2}{e^{6\beta}} + \frac{3q^3}{e^{8\beta}} + \frac{4q^4}{e^{10\beta}} + \cdots$, algebraically evidenced in

$$\frac{e^{2in}}{n^3(e^{2in}-e^{2\beta})^2} = \frac{q}{n^3(q-e^{2\beta})^2} = \frac{q}{n^3e^{4\beta}} + \frac{2q^2}{n^3e^{6\beta}} + \frac{3q^3}{n^3e^{8\beta}} + \frac{4q^4}{n^3e^{10\beta}} + \cdots, \quad (20)$$

as a result, I establish $Ł_z(\beta)$ as below

$$Ł_z(\beta) = -4e^{2\beta}\sum_{n=1}^{\infty}\left\{\frac{q}{n^3e^{4\beta}} + \frac{2q^2}{n^3e^{6\beta}} + \frac{3q^3}{n^3e^{8\beta}} + \cdots\right\}$$
$$= -4e^{2\beta}\{\sum_{n=1}^{\infty}\frac{q}{n^3e^{4\beta}} + \sum_{n=1}^{\infty}\frac{2q^2}{n^3e^{6\beta}} + \sum_{n=1}^{\infty}\frac{3q^3}{n^3e^{8\beta}} + \cdots\}$$
$$= -4e^{2\beta}\{\frac{1}{e^{4\beta}}\sum_{n=1}^{\infty}\frac{q}{n^3} + \frac{2}{e^{6\beta}}\sum_{n=1}^{\infty}\frac{q^2}{n^3} + \frac{3}{e^{8\beta}}\sum_{n=1}^{\infty}\frac{q^3}{n^3} + \cdots\}. \quad (21)$$

Particularly, in the next step, I prove that each of the sums given by $\sum_{n=1}^{\infty}\frac{q^k}{n^3}$ in (21), $k = 1, 2, 3, \ldots$, can be computed using the polylogarithm $Li_3(e^{i2k})$, with $k = 1, 2, 3\ldots$, because the definition of $Li_s(z)$, [14], is

$$Li_s(z) = \sum_{n=1}^{\infty} n^{-s} z^n, \quad (22)$$

and I just replace $s = 3$ and $z = q = e^{2ik}$, with $k = 1, 2, 3, \ldots$, in order to get

$$Li_3(e^{2ik}) = \sum_{n=1}^{\infty} n^{-3}(e^{2ik})^n = \sum_{n=1}^{\infty}\frac{e^{2ikn}}{n^3}. \quad (23)$$

I use the relationship (23) to conveniently achieve a *polylogarithmic expansion of* $Ł_z(\beta)$ as a never seen additional finding within the context of the Flint-Hills series.

Thus, *the Flint-Hills-López series* $Ł_z(\beta)$ is equivalent to

$$\boldsymbol{Ł_z(\beta) = -4e^{2\beta}\left\{\frac{1}{e^{4\beta}}Li_3(e^{2i}) + \frac{2}{e^{6\beta}}Li_3(e^{4i}) + \frac{3}{e^{8\beta}}Li_3(e^{6i})\ldots\right\} = \frac{-4e^{2\beta}}{e^{4\beta}}\sum_{k=1}^{\infty} k\frac{Li_3(e^{2ik})}{e^{2k\beta-2\beta}}}.$$

$$Ł_z(\beta) := \sum_{n=1}^{\infty}\frac{csc^2(n+i\beta)}{n^3} \equiv -4\sum_{k=1}^{\infty} k\frac{Li_3(e^{2ik})}{e^{2k\beta}},$$

$$Ł_z(\beta) := -4\sum_{k=1}^{\infty} k\,e^{-2k\beta}Li_3(e^{2ik}), \quad (24)$$

so I call (24) *The López-Flint-Hills Polylogarithmic Series Expansion for* $Ł_z(\beta)$. The equation (24) converges within the complex analysis for every real value β except zero. However, using *Definition 1*, the complete depicted scenario for the Flint Hills is

$$Ł_z(\beta) := \begin{cases} \tau - \sigma = 30.3144204831\ldots, \; if\ \beta = 0 \\ or \\ -4\sum_{k=1}^{\infty} k\,e^{-2k\beta}Li_3(e^{2ik}), \; if\ \beta\ is\ real \end{cases}, \quad (25)$$

VII. Definition of $Ł_z(\beta)$ Based on the Integral of the Bose-Einstein Distribution

I notice the prevalence of a similar polylogarithm $Li_3(e^{2ik})$ from a general version of the integral of the Bose-Einstein distribution that includes the complex argument e^{2ik} instead of the real version of e^{ε_B}, [15], with ε_B a non-complex number, in some applications of physics, or also a general complex number e^z, [16], which is similar to e^{2ik} in $Li_3(e^{2ik})$ through the expressions (26) and (27)

$$Li_{s+1}(z) = \frac{1}{\Gamma(s+1)}\int_0^{\infty}\frac{t^s}{\frac{e^t}{z}-1}dt, \quad (26)$$

where $Re(s) > 0$. I notice that at the values $s = 2$ and $z = e^{2ik}$ *the Bose-Einstein-like integral* leads to represent $Li_3(e^{2ik})$ as

$$Li_3(e^{2ik}) = \frac{1}{\Gamma(2+1)}\int_0^{\infty}\frac{t^2}{(\frac{e^t}{e^{2ik}}-1)}dt =$$

$$\frac{1}{\Gamma(3)}\int_0^{\infty}\frac{t^2}{(e^{t-2ik}-1)}dt=\frac{1}{2!}\int_0^{\infty}\frac{t^2}{(e^{t-2ik}-1)}dt, \quad (27)$$

thus, I can establish a correspondence between that definition in (27) and the *López-Flint-Hills Polylogarithmic Series Expansion for* $Ł_z(\beta)$

$$Ł_z(\beta)=-4\sum_{k=1}^{\infty}k\,e^{-2k\beta}[\frac{1}{2!}\int_0^{\infty}\frac{t^2}{(e^{t-2ik}-1)}dt]=$$
$$-2\int_0^{\infty}t^2\sum_{k=1}^{\infty}k\frac{e^{-2k\beta}}{(e^{t-2ik}-1)}dt, \quad (28)$$

being (28) *the Bose-Einstein-like integral representation of the Flints-Hills-López series* $Ł_z(\beta)$. A careful revision of (28) could lead to exploring some links between important concepts in physics, e.g., condensate-matter states, behind the nature of the Flint-Hills series. For achieving that link, it would be necessary to understand if there would be a crucial role for the variable t in (28) or a modified version of $Ł_z(\beta)$ within the analysis of physics because it is already known that the Bose-Einstein distribution describes the statistical behavior of integer spin particles (bosons), [17]. Of course, in this paper, I am just opening the door to new possibilities in this research. As a resemblance between (28) for the case of $Ł_z(\beta)$ and the Bose-Einstein distribution for a sys-tem of iden-ti-cal bosons, [18], compare (29) to (28)

$$\natural^b=\frac{1}{e^{(E^p-\mu)/K_BT}-1}, \quad (29)$$

or also to (30) which is another extended definition of the Bose-Einstein distribution, [19],

$$P(k)=\frac{k^s}{e^{(k-\mu)}-1}, \quad (30)$$

where $\frac{t^2}{(e^{t-2ik}-1)}$ from (27) is similar to $P(k)$ at $s=2$, of course, by just switching the name of the variables used, i.e., k by t, and extending μ to the imaginary domain as $\mu=2ik$. This proposal establishes an extraordinary similarity between the shape of the distribution given by $\natural^b$ or $P(k)=P(t)$ within an unexplored context for the problem of the Flint-Hills or its generalizations. For clarifying the terms in (29) and (30), here $\natural^b$ is the av-er-age num-ber of bosons in a sin-gle-par-ti-cle state with sin-gle-par-ti-cle en-ergy E^p. Fur-ther T is the ab-solute tem-per-a-ture, and K_B is the Boltz-mann con-stant, [18], equal to $1{,}38065x10^{-23}$ J/K. The integral of the distribution (30) is related to the Bose-Einstein-like integral I have exposed in the current section, as a result below

$$\int_0^{\infty}\frac{k^s}{e^{(k-\mu)}-1}dk=\Gamma(s+1)Li_{s+1}(e^{\mu})\rightarrow resemblance\ of\rightarrow$$
$$\Gamma(s+1)Li_{s+1}(z)=\int_0^{\infty}\frac{t^s}{\frac{e^t}{z}-1}dt, \quad (31)$$

and

$$\int_0^{\infty}\frac{k^2}{e^{(k-\mu)}-1}dk=\Gamma(3)Li_3(e^{\mu})\rightarrow resemblance\ of\rightarrow$$

$$\Gamma(3)Li_3(e^{2ik})=\int_0^{\infty}\frac{t^2}{e^{t-2ik}-1}dt. \quad (32)$$

VIII. Proof of Convergence of the Cookson Hills Series based on the Adamchik-Srivastava Summation Formula

Basically, the proof is similar to the case of the Flint Hills series, i.e., it is possible to find two convergent series that are bounded and never diverge and lead to represent the Cookson Hills series. For that purpose, let $t=\frac{1}{\pi^2}e^{-2sec^2(n)}$ be replaced in $\sum_{m=1}^{\infty}\frac{t^m}{m^2}\zeta(2m)=log(\pi\sqrt{t}\,csc(\pi\sqrt{t}\,))$ because the term $sec^2(n)$ appears in the definition of the Cookson Hills series $\sum_{n=1}^{\infty}\frac{sec^2(n)}{n^3}$ and can be modeled as follows

$$\sum_{m=1}^{\infty}\frac{e^{-2m\,sec^2(n)}}{\pi^{2m}m^2}\zeta(2m)=$$
$$log(\pi\sqrt{\frac{1}{\pi^2}e^{-2sec^2(n)}}\,csc(\pi\sqrt{\frac{1}{\pi^2}e^{-2sec^2(n)}})),$$

then, I get

$$\sum_{m=1}^{\infty}\frac{e^{-2m\,sec^2(n)}}{\pi^{2m}m^2}\zeta(2m)=-sec^2(n)\ +log\ csc(e^{-sec^2(n)}).$$

Now, divide by n^3 in both sides of the previous expression, I obtain

$$\frac{1}{n^3}\sum_{m=1}^{\infty}\frac{e^{-2m\,sec^2(n)}}{\pi^{2m}m^2}\zeta(2m)=-\frac{1}{n^3}sec^2(n)\ +$$
$$\frac{1}{n^3}log\ csc(e^{-sec^2(n)}),$$

and just establishing the sum

$$\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{m=1}^{\infty}\frac{e^{-2m\,sec^2(n)}}{\pi^{2m}m^2}\zeta(2m)=-\sum_{n=1}^{\infty}\frac{1}{n^3}sec^2(n)+$$
$$\sum_{n=1}^{\infty}\frac{1}{n^3}log\ csc(e^{-sec^2(n)})\ ,$$

which leads to represent the Cookson-Hills series as follows

$$\sum_{n=1}^{\infty}\frac{sec^2(n)}{n^3}=\sum_{n=1}^{\infty}\frac{1}{n^3}log\ csc(e^{-sec^2(n)})-$$
$$\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{m=1}^{\infty}\frac{e^{-2m\,sec^2(n)}}{\pi^{2m}m^2}\zeta(2m). \quad (33)$$

The relationship (33) also converges thanks to the series

$$S_{C_1}=\sum_{n=1}^{\infty}\frac{1}{n^3}log\ csc(e^{-sec^2(n)}),$$

and $S_{C_2}=\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{m=1}^{\infty}\frac{e^{-2m\,sec^2(n)}}{\pi^{2m}m^2}\zeta(2m)$, being S_{C2} *the Minor Zeta Cookson-Hills-López series.* It is practically the same methodology used in the scenario of the convergence of the Flint Hills series because $sec^2(n)$ is related to similar behavior of abrupt jumps perceived in the trigonometric function $csc^2(n)$. That is why the Adamchik-Srivastava summation formula could serve to generalize several cases based on trigonometric functions with abrupt nature.

Definition 4. The Cookson-Hills-López series representation for $\sum_{n=1}^{\infty}\frac{sec^2(n)}{n^3}$ is given by subtracting two convergent series S_{C_1} and S_{C_2} derived from the analysis of the Adamchik-Srivastava Summation Formula used previously, i.e., (1),

$$\sum_{m=1}^{\infty}\frac{t^m}{m^2}\zeta(2m) = log(\pi\sqrt{t}\, csc(\pi\sqrt{t}\,)),$$

when $t = \frac{1}{\pi^2}e^{-2sec^2(n)}$, *such that* S_{C_1} *converges to* $\varsigma = 42.9960\ ...$ *and* S_{C_2} *converges to* $\varrho \approx 0.0012$. *The Cookson Hills series obeys the convergent value* $\varsigma - \varrho = 42.994\ ...$

$$\sum_{n=1}^{\infty}\frac{sec^2(n)}{n^3} = S_{C_1} - S_{C_2} = \varsigma - \varrho = 42.994\ ...,$$

where

$$S_{C_1} = \sum_{n=1}^{\infty}\frac{1}{n^3}log\ csc(e^{-sec^2(n)}) = \varsigma = 42.9960\ ...,$$

$$S_{C_2} = \sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{m=1}^{\infty}\frac{e^{-2m\,sec^2(n)}}{\pi^{2m}m^2}\zeta(2m) = \varrho \approx 0.0012.$$

Lemma 4. The sequence $\{\mu(n) = \frac{1}{n^3}log\ cs\,c\ e^{-sec^2(n)}\}$, *with* $n = 1, 2, 3, ...$, *is bounded and achieves the definition of bounded sequences given by*

there exists a real number $M = 38.35821564\ ...$ *such that* $|\mu(n)| \leq M$ *for all natural numbers* N, $|\ n = 1, 2, 3, ...$ *Thus, the sequence is bounded as* $0 < |\mu(n)| \leq 38.35821564\ ...$ *being* $\mu(n)$ *always positive and decreasing.*

Proof. The plot, given in Fig.7, of every sample $\mu(n) = \{\mu_{(1),}\mu_{(2),}\mu_{(3),}\ ...\ \};\ |\ \mu(n) = lo\,g\ \{\,csc\big(e^{-sec^2(n)}\big)\,\}^{n^{-3}}$, shows a slow-damped behavior regarding $\mu(n)$. The highest sample of the whole set is the computed value $\mu(11) = M = 38.3582156\ ...$ so the rest of the values are decreasing and eventually get vanished in the infinite as the value $n \to \infty$.

The routine in Wolfram Alpha, Fig. 6, computes in less than 1 minute for $n = 9000$ samples the expected main value seen in the plots of some references like in [6]. Anybody can calculate the sum S_{C_1} that converges to $\varsigma = 42.9960\ ...$ by the command line of Natural Language used in Wolfram Alpha below

*sum (1/(n^3)*log(csc(1/exp(sec(n)^2)))) from n=1 to 9000*

Figure 6. Computation of the main sum

$S_{C_1} = \sum_{n=1}^{\infty}\frac{1}{n^3}\log\csc(e^{-\sec^2(n)})$ via Wolfram Alpha.

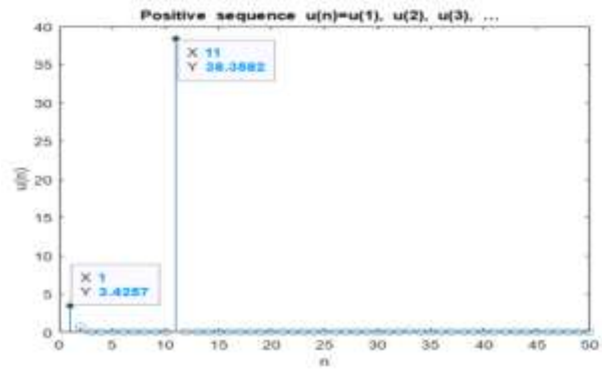

Figure 7. Plot of the bounded sequence μ(n) showing the highest values μ(1) = 3.4257... and M = μ(11) = 38.3582.. Cookson Hills series.

Lemma 5. The series

$S_{C_2} = \sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{m=1}^{\infty}\frac{e^{-2m\,sec^2(n)}}{\pi^{2m}m^2}\zeta(2m) = \varrho \approx 0.0012$ *which happens because the term* $\frac{e^{-2m\,sec^2(n)}}{\pi^{2m}m^2}$ *affects quickly the nature of convergence causing these series to be bounded and finite.*

Proof. The successive partial sums were given by $\frac{1}{n^3}\sum_{m=1}^{\infty}\frac{e^{-2m\,sec^2(n)}}{\pi^{2m}m^2}\zeta(2m)$ per each $n = 1, 2, 3, ...$, show behavior in a similar way that in the case of the Flint Hills series studied before. The first partial sums can be computed to calculate a valid approximate value of convergence given by the result

$S_{C_2} \approx 0.000176396 + 2.00992\,x\,10^{-7} + 0.00080388 + 0.0000241413 + 0.000178103 + 0.00010237 + \cdots . +0 + 0 + \cdots \approx 0.0012.$

As numerical evidence for these partial sums associated to S_{C_2}, computing higher samples like, for example, $n = 100$ in radians, by the code in Wolfram Alpha given by

*Sum[Divide[ζ\(40)2m\(41)Power[100,-3],Square[\(40)m*Power[π,m]\(41)]exp\(40)2*m*Square[\(40)sec\(40)100\(41)\(41)]\(41)],{m,1,∞}]*

make clear the natural tendency of the higher partial sums of S_{C_2} to get vanished or practically equal to zero in the infinite, $n \to \infty$, due to the prevalence of $\frac{e^{-2m\,sec^2(n)}}{\pi^{2m} m^2}$ that decays easily to zero. Therefore, the first one hundred samples between $n = 1$ and $n = 100$ are quiet enough for getting a precise approximation of S_{C_2} because the series $S_{C_2} = \sum_{n=1}^{\infty} \frac{1}{n^3} \sum_{m=1}^{\infty} \frac{e^{-2m\,sec^2(n)}}{\pi^{2m} m^2} \zeta(2m)$ can achieve a quick result on convergence but a little slow compared to the Minor Zeta Flint-Hills-López series of the Flint Hills series, which is quicker to reach partial sums that converge to the effective value for that series. Nevertheless, The Minor Zeta Cookson-Hills-López series S_{C_2} has a small contribution $\varrho \approx 0.0012$ but must be included; all are consistent with the known plots in the references based on the pure expansion of the original series, [6].

IX. The Apéry constant and its Relationship with the Cookson Hill Series and the Convergent Series $\sum_{n\geq 1} \frac{tan^2(n)}{n^3}$

I derive a second new representation for the Cookson Hills series based on the well known

$$sec^2(\theta) = 1 + tan^2(\theta), \tag{34}$$

with an integer $\theta = n$, being $n = 1, 2, 3, \ldots$. I multiply both sides of (34) by $\frac{1}{n^3}$ and arrange (34) as

$$\frac{sec^2(n)}{n^3} - \frac{tan^2(n)}{n^3} = \frac{1}{n^3}, \tag{35}$$

again, $n \neq 0$, within this context. Then, I let $\{\frac{sec^2(n)}{n^3} :\ n \geq 1\}$ be the sequence related to the Cookson Hills series $\sum_{n\geq 1} \frac{sec^2(n)}{n^3}$ and $\{\frac{tan^2(n)}{n^3} :\ n \geq 1\}$ a new sequence derived from the second term $\frac{tan^2(n)}{n^3}$ seen on the left side of (35). Therefore, I establish a convergent series given by

$$\sum_{n\geq 1} \frac{tan^2(n)}{n^3}, \tag{36}$$

again I get a resembling representation of the Apéry constant by the definition below

Definition 5. The López Cookson-Hills representation involving the Apéry constant is given by the relations below

$$\zeta(3) = \sum_{n\geq 1} \frac{sec^2(n)}{n^3} - \sum_{n\geq 1} \frac{tan^2(n)}{n^3},$$

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{log\ csc\left(e^{-sec^2(n)}\right)}{n^3} - \sum_{n=1}^{\infty} \frac{1}{n^3} \sum_{m=1}^{\infty} \frac{e^{-2m\,sec^2(n)}}{(m\,\pi^m)^2} \zeta(2m)$$

$$- \sum_{n=1}^{\infty} \frac{1}{n^3} log\ csc(e^{-tan^2(n)})$$

$$+ \sum_{n=1}^{\infty} \frac{1}{n^3} \sum_{m=1}^{\infty} \frac{e^{-2m\,tan^2(n)}}{(m\,\pi^m)^2} \zeta(2m).$$

That representation is not a mystery considering that the Pythagorean identities used in this analysis link the summations to the Apéry constant $\zeta(3)$. Therefore, $\sum_{n\geq 1} \frac{sec^2(n)}{n^3}$ can be represented by

$$\sum_{n\geq 1} \frac{sec^2(n)}{n^3} = \zeta(3) + \sum_{n=1}^{\infty} \frac{log\ csc(e^{-tan^2(n)})}{n^3}$$

$$- \sum_{n=1}^{\infty} \frac{1}{n^3} \sum_{m=1}^{\infty} \frac{e^{-2m\,tan^2(n)}}{(m\,\pi^m)^2} \zeta(2m). \tag{37}$$

X. A Novel Representation of the Cookson Hills Series as the Difference of two Flint Hills series

I prove that there exists a novel representation of the Cookson Hills series given by

$$\sum_{n\geq 1} \frac{sec^2(n)}{n^3} = 4 \sum_{n\geq 1} \frac{csc^2(2n)}{n^3} - \sum_{n\geq 1} \frac{csc^2(n)}{n^3}. \tag{38}$$

I just use the trigonometric identity $csc^2(n) = \frac{sec^2(n)}{sec^2(n)-1}$ which can be manipulated successively as

$$sec^2(n) = csc^2(n) sec^2(n) - csc^2(n),$$

$$sec^2(n) = \frac{1}{sin^2(n)\,cos^2(n)} - csc^2(n),$$

$$sec^2(n) = \frac{1}{(\frac{2\,sin(n)\,cos(n)}{2})^2} - csc^2(n),$$

and recognizing the term $2\,sin(n)\,cos(n) = sin(2n)$, I write then

$$sec^2(n) = \frac{1}{(\frac{sin(2n)}{2})^2} - csc^2(n) = \frac{4}{(\,sin(2n)\,)^2} - csc^2(n),$$

which is divided by n^3 as follows

$$\frac{sec^2(n)}{n^3} = \frac{4}{n^3(\,sin(2n)\,)^2} - \frac{csc^2(n)}{n^3},$$

showing, as clearly visible, the Cookson Hills series because the expression

$\frac{4}{n^3(\,sin(2n))^2} = \frac{4}{n^3} csc^2(2n)$ denotes the main term, leading to represent the Cookson Hills series as exposed below

$$\sum_{n\geq 1} \frac{sec^2(n)}{n^3} = 4 \sum_{n\geq 1} \frac{csc^2(2n)}{n^3} - \sum_{n\geq 1} \frac{csc^2(n)}{n^3}.$$

Hence, relationship (38) means that the difference between four times the modified Flint Hills series $\sum_{n\geq 1}\frac{csc^2(2n)}{n^3}$, which is a generic case when $f = 2$ in $\sum_{n\geq 1}\frac{csc^2(f\,n)}{n^3}$, and the original version $\sum_{n\geq 1}\frac{csc^2(n)}{n^3}$ let us represent the problem of the Cookson Hills series as established in the known references. Therefore, (38) is highlighted as a finding to compute rather the Flint Hill series modified by $f = 2$ in $\sum_{n\geq 1}\frac{csc^2(f\,n)}{n^3}$, i.e.,

$$\sum_{n\geq 1}\frac{csc^2(fn)}{n^3} = \frac{1}{4}\sum_{n\geq 1}\frac{sec^2(n)}{n^3} + \frac{1}{4}\sum_{n\geq 1}\frac{csc^2(n)}{n^3}, \quad (39)$$

where $f = 2$.

Lemma 6. The general case $\sum_{n\geq 1}\frac{csc^2(f\,n)}{n^3}$, when $f = 2$ in the Flint Hills series, converges to

$$\sum_{n\geq 1}\frac{csc^2(fn)}{n^3} = \frac{1}{4}(\varsigma - \varrho) + \frac{1}{4}(\tau - \sigma) = \frac{1}{4}(42.994) + \frac{1}{4}(30.3144204831) \approx 18.32710. \quad (40)$$

Proof. Based on the relationships already obtained for the Flint Hills series when $f = 1$, i.e., $\sum_{n\geq 1}\frac{csc^2(fn)}{n^3} = \tau - \sigma = 30.3144204831$ and the Cookson Hills series $\sum_{n\geq 1}\frac{sec^2(f\,n)}{n^3}$ when $f = 1$, the substitution in (40) leads to computing the series $\sum_{n\geq 1}\frac{csc^2(2n)}{n^3}$ which constitutes another important finding that let us establish a future analysis of convergence for advanced general cases when $f > 2, f = 2, 3, 4, ...$ The evidence establishes a fascinating path to approach several unknown series.

XI. Discussion about this Research

Having established the convergence of the Flint Hills and Cookson Hills series, the focus now shifts to understanding the usefulness of the Adamchik-Srivastava summation formula in achieving convergence for series involving trigonometric functions. The findings of this research reveal alternative series representations that effectively solve the mystery of convergence. This discussion is crucial in shedding light on the underlying principles and potential criteria for convergence.

The fundamental aspect of this discovery lies in the ability to transform a series with erratic behavior, as demonstrated by the jumps resulting from the cosecant and secant functions, into a stable series. This transformation can be achieved by finding a formula that organizes the sequences in a manner that ensures positive and bounded terms, thereby avoiding infinite values. This aspect holds significant importance as it establishes the groundwork for a future criterion applicable to the general cases of the Flint Hills and Cookson Hills series, as well as their variations.

The application of such a criterion holds potential in solving physical problems where models exhibit series with abrupt behavior. Physics or other sciences must remain consistent in its treatment of series solutions, ensuring the absence of infinite or abrupt jumps in the final solutions observed. By transforming the series into bounded systems with positive terms mainly, the localization of any remaining infinite terms becomes crucial. The future criterion, based on the Adamchik-Srivastava summation formula, can be extended to incorporate similar formulas involving the Riemann Zeta function to precisely identify and localize these specific infinities or demonstrate their absence.

Convergence of a series implies that terms of sequences can be expressed in a positive and bounded manner, thereby avoiding peculiar infinite behaviors. However, if a series, similar to the Flint Hills series, were to transform using a new formula resulting in the representation of stability, while unequivocally showing the absence of any appropriate transformation to deliver finite bounded sequences, then it can be concluded that the series is non-convergent. In such cases, methods to represent these inherently divergent series would be unavailable according to the context of the use of such series. The tasks pursued in this article successfully addressed the essence of convergence of the Flint Hills and Cookson Hills series, providing clarity and insight into the behavior of these series.

In conclusion, this discussion highlights the significance of the Adamchik-Srivastava summation formula in achieving convergence for series involving trigonometric functions. The findings presented in this research open up avenues for future exploration, both in terms of extending the criterion to encompass broader cases and utilizing similar formulas to locate and analyze infinities within series representations. The tasks accomplished in this article have successfully unraveled the essential aspects related to the convergence of the Flint Hills and Cookson Hills series and provide a valuable contribution to the field of these. In the future, the discussion about if these general series $\sum_{n\geq 1}\frac{csc^2(f\,n)}{n^3}$, $\sum_{n\geq 1}\frac{sec^2(f\,n)}{n^3}$, or even the relevant cases $\sum_{n\geq 1}\frac{csc^v(f\,n)}{n^u}$ and $\sum_{n\geq 1}\frac{sec^v(f\,n)}{n^u}$, for several pairs (u, v) is a matter of serious research as basically the fundamental cases $\sum_{n\geq 1}\frac{csc^2(n)}{n^3}$, $\sum_{n\geq 1}\frac{csc^2(2n)}{n^3}$ and $\sum_{n\geq 1}\frac{sec^2(n)}{n^3}$ were proved to converge and their observation should lead to finding a method that generalized all cases. This paper contributes to offering a way to get more clear criteria in the close future for these special series.

In light of the implications of the nature of the number π and the newly established upper bound for its irrationality measure, significant insights emerge regarding the continuity and integrability of the one-dimensional surface area of the circle. Contrary to prevailing beliefs concerning a unit circle, the convergence of the Flint Hills series, facilitated by the aforementioned formula, implies a revised upper bound of $\mu(\pi) \leq 5/2$ for the number π. This revised bound effectively governs the essential irrationality of π and subsequently leads to a reduction in the integrability of the one-dimensional surface area of the unit circle. The profound significance of this discovery becomes evident through the proof of the Flint Hills series, revealing the profound impact of the irrationality of the number π on geometry. The degree to which π is irrational directly influences the underlying geometry, underscoring the far-reaching implications of this result.

Also, it is noteworthy that the Apéry constant is an outcome of investigating the properties of the Flint Hills series through this representation

$$\zeta(3) = \sum_{n\geq 1} \frac{sec^2(n)}{n^3} - \sum_{n\geq 1} \frac{tan^2(n)}{n^3}.$$

In this discussion, I present also the hypothesis that due to the nature of the Riemann Zeta function there exist values related to the Bernoulli numbers $B_4 = -1/30$ and $B_6 = 1/42$ in the integer parts of the convergence of the Flint Hills series seen as

$$\sum_{n=1}^{\infty} \frac{csc^2(n)}{n^3} = \frac{-1}{B_4} + 0.3144204831 \ldots = 30 + 0.31442\ldots$$

and

$$\sum_{n\geq 1} \frac{\sec^2 n}{n^3} = \frac{1}{B_6} + 0.994 \ldots = 42 + 0.994 \ldots = 42.994 \ldots.$$

A precise convergence of the formulas, in the future, in a more advanced analysis, should demonstrate that the Bernoulli numbers $B_4 = -1/30$ and $B_6 = 1/42$ through their inverses are, in fact, the integer parts evidenced like the problems of the Flint Hills and Cookson Hills series.

XII. Conclusion

In conclusion, this paper has successfully demonstrated the convergence of the Flint Hills series to the expected value of 30.3144... and the convergence of the Cookson Hills series to 42.994... These results align with the statistical findings and provide strong evidence for the accuracy of the computed values. The analysis and methodology presented in this study have shed light on the behavior and convergence of these series, offering valuable insights into their mathematical properties. These findings contribute to a deeper understanding of the Flint Hills and Cookson Hills series, as well as their relevance in the broader context of number theory and mathematical analysis, e.g., in the definition of the new upper bound for the irrationality measure of π, i.e., $\mu(\pi) \leq 2.5$ and the implications in the area of a unit circle as discussed. The confirmed convergence of these series serves as a crucial benchmark for future research and opens up avenues for exploring their applications. Moreover, the perspectives for the Flint Hills and Cookson Hills series are addressed to solving particular cases for the pairs $(u, v) = (7, 1)$, (14, 2), (20, 3), and for higher cases, and to prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^u |sin(n)|^v}$ converges for $(u, v) = (8, 1)$, (15, 2), (21, 3), etc.,. It is a pending work in this research.

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