Sub JDB-semigroup, JD-field, and JD-ideal

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Abstract: This paper introduces the notion of the JDB-semigroup, an extended study of dual B-algebra by applying the concept of semigroup. Some properties and characteristics of sub JDB-semigroup, units, unity, JD-field, and JD-ideal in a JDB-semigroup are presented in this study.

Keywords: Dual B-algebra, JD-ideal, JD-field, JDB-semgiroup.

I. Introduction

In [7], introduced the notion of B-algebra which is related to several classes of algebras such as BCH/BCI/BCK-algebras. A B-algebra is a triple $(X, *, 0)$ where X is a nonempty set, $*$ as a binary operation on X and a constant 0 such that it satisfies the following axioms:

B1. $x * x = 0$

B2. $x * 0 = x$

B3. $(x * y) * z = x * (z * (0 * y))$

In [7], the authors demonstrates an interesting relationship between B-algebras and groups.

In [2], introduced and characterized the notion of dual B-algebra. A dual B-algebra is a nonempty set X and a constant 1 and a binary operation $\circ$ such that it satisfies the following axioms for all x, y, $z \in X$:

DB1. $x \circ x = 1$

DB2. $1 \circ x = x$

DB3. $x \circ (y \circ z) = ((y \circ 1) \circ x) \circ z$

The study of dual B-algebra investigates the relationship between dual B-algebra and BCK-algebra and provides some of its initial properties, the study also presents the relationship between dual B-algebra and B-algebra. They also discussed the commutativity of a dual B-algebra and its relationship to other algebra such as CI-algebra and dual BCI-algebra. In [2], they proved that every B-algebra determines a dual B-algebra called the *derived dual B-algebra*.

In [4], introduced and investigated the KS-semigroup which is related to BCK-algebras and semigroups. In their paper, they introduced the ideal of KS-semigroups and a strong KS-semigroup, some characterization of ideals of KS-semigroups are also provided. In [6], introduced the notion of JB-semigroup, a new algebra that incorporates the concept of semigroup into B-algebra. An algebra $(X, *, \cdot, 0)$ is called a JB-semigroup if it satisfies the following:

i. $(X, *, 0)$ is a B-algebra;

ii. $(X, \cdot)$ is a semigroup;

iii. The operation $\cdot$ is left and right distributive over the operation $*$.

The study of JB-semigroup proved that every ring determines a JB-semigroup, but the converse need not to be true. In their paper, they define JB-field and JB-domain and prove that every JB-field is a JB-domain and every finite JB-domain is a JB-field. In addition, the concept of JB-ideal of JB-semigroup was presented, and the qoutient JB-semigroup was constructed using JB-ideal, as well as some of its properties. In addition, the unity in a JB-semigroup was introduced and denoted by 1. Related properties of unity and 1-invertible elements are also discussed.

The study of KS-semigroup and JB-semigroup implies that the idea of a semigroup can be usefully incorporated into a wide variety of algebraic topics. With the aforementioned studies, this research provides the evidence for the existence of semigroup within the dual B-algebra and outline the structure of JDB-semigroup. This paper also shows that the JDB-semigroup has specific properties concerning the dual B-algebra. Specifically, some properties of the sub JDB-semigroup, units, unity, JD-field, and JD-ideal of a JDB-semigroup are also provided in this study.

II. Preliminaries

Definition 1. [5] A *binary operation* "$*$" on a set S is a function mapping $S \times S$ into S. For each $(a, b) \in S \times S$, we will denote the element $*((a, b))$ of S by $a * b$.

Theorem 1. [3] *Let S be a nonempty subset of a dual B-algebra X. Then S is a dual B-subalgebra if and only if for any $x, y \in S$, $x \circ y \in S$.*

Theorem 2. [2] Let $X = (X, \circ, 1)$ be any algebra of type $(2, 0)$. Then X is a dual B-algebra if and only if for any $x, y, z \in X$.

i. $(x \circ y) \circ (x \circ z) = y \circ z$

Lemma 1. [2] *Let X be a dual B-algebra. Then for any $x, y, z \in X$, we have*

i. $(x \circ 1) \circ 1 = x$

ii. $(y \circ z) \circ x = z \circ [(y \circ 1) \circ x]$

Remark 1. [2] *If $(X, *, 0)$ is a B-algebra, define "$\circ$" as follows: $x \circ y = y * x$ for all $x, y \in X$. Then $(X, \circ, 0)$ is a dual B-algebra, called the derived dual B-algebra.*

Remark 2. [2] *Not every dual B-algebra is a B-algebra and not every B-algebra is a dual B-algebra.*

Definition 2. [1] A *semigroup* is an ordered pair of the form $(G, \cdot)$ where G is a set and $\cdot$ is an associative binary operation on G.

III. Sub JDB-semigroup, JD-field, and JD-ideal

Definition 3. A *JDB − semigroup* is a quadruple $(X, \circ, \cdot, 1)$ where X is a nonempty set, "$\circ$" and "$\cdot$" are the binary operations on X, and a constant 1 such that the following axioms are satisfied for all x, y, z in X:

JD1. $(X, \circ, 1)$ is a dual B-algebra;

JD2. $(X, \cdot)$ is a semigroup; and

JD3. The operation "$\cdot$" is left and right distributive over the operation "$\circ$".

It follows from Definition 3 that if $(X, \circ, \cdot, 1)$ is a JDB-semigroup, then all characteristics associated with the binary operation $\circ$ with respect to the dual B-algebra $(X, \circ, 1)$ also hold for the JDB-semigroup.

In the study of [2], every B-algebra determines a dual B-algebra called the derived dual B-algebra (See Remark 1). The next remark describes the relationship between JB-semigroup and JDB-semiroup.

Remark 3. *If $(X, *, \cdot, 0)$ is a JB-semigroup, define "$\circ$" as $x \circ y = y * x$. Then $(X, \circ, \cdot, 1)$ is a JDB-semigroup called the derived JDB-semigroup.*

In [2], the authors also proved that not every dual B-algebra is a B-algebra and not every B-alebra is a dual B-algebra, it easily follows that not every JDB-semigroup is a JB-semigroup and not every every JB-semigroup is a JDB-semigroup.

Below is an example of a JDB-semigroup.

Example 1. Let $X = \{1, a, b, c\}$ with the following tables:

$\circ$	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

$\cdot$	1	a	b	c
1	1	1	1	1
a	1	a	b	c
b	1	b	c	a
c	1	c	a	b

By routine computations, $(X, \circ, \cdot, 1)$ is a JDB-semigroup.

The following example shows that the set complex numbers is not a JDB-semigroup.

Example 2. Let $X = \mathbb{C}$ be the set of complex numbers. Define $\circ$ as $a \circ b = \dfrac{b}{a}$ for all $a, b \in X$, with $a \neq 0$ and $\cdot$ be the usual multiplication. Thus, $(X, \circ, 1)$ is a dual B-algebra but not a JDB-semigroup.

Solution: Suppose $a, b, c \in \mathbb{C}$ such that $a = x + iy$, $b = u + iv$, $c = r + is$. Note that $(X, \circ, 1)$ satisfies (DB1): $a \circ a = (x + iy) \circ (x + iy) = \dfrac{x + iy}{x + iy} = 1$, (DB2): $1 \circ a = \dfrac{x + iy}{1} = x + iy = a$, and (DB3): $a \circ (b \circ c) = \dfrac{\frac{r+is}{u+iv}}{x + iy} = \dfrac{r + is}{(u + iv)(x + iy)} = \dfrac{r + is}{\frac{x+iy}{\frac{1}{u+iv}}} = ((b \circ 1) \circ a) \circ c$. Thus, $(X, \circ, 1)$ is a dual B-algebra. Since $\cdot$ is associative, then $(X, \cdot)$ is a semigroup. Observe that by JD3, $a \cdot (b \circ c) = a \cdot \dfrac{c}{b} = \dfrac{ac}{b} \neq \dfrac{c}{b} = \dfrac{ac}{ab} = (a \cdot b) \circ (a \cdot c)$. Therefore, $X = \mathbb{C}$ is a dual B-algebra but not a JDB-semigroup.

This example leads to the following remark.

Remark 4. *A dual B-algebra with an associative operation is not always a JDB-semigroup.*

The following properties also hold using the derived JDB-semigroup.

Lemma 2. *Let X be a JDB-semigroup $(X, \circ, \cdot, 1)$. Then for all $a, b, c \in X$,*

i. $a \cdot 1 = 1 \cdot a = 1$,

ii. $a \cdot (b \circ 1) = (a \circ 1) \cdot b = (a \cdot b) \circ 1$,

iii. $(a \circ 1) \cdot (b \circ 1) = a \cdot b$,

iv. $a \cdot ((c \circ 1) \circ b) = ((a \cdot c) \circ 1) \circ (a \cdot b)$, $((c \circ 1) \circ b) \cdot a = ((c \cdot a) \circ 1) \circ (b \cdot a)$.

Proof: Let $a, b, c \in X$.

i. By DB1 and JD3, $a \cdot 1 = a \cdot (1 \circ 1) = (a \cdot 1) \circ (a \cdot 1) = 1$. Similarly, $1 \cdot a = (1 \circ 1) \cdot a = (1 \cdot a) \circ (1 \cdot a) = 1$.

ii. By JD3 and (i), $a \cdot (b \circ 1) = (a \cdot b) \circ (a \cdot 1) = (a \cdot b) \circ 1$. Similarly, $(a \circ 1) \cdot b = (a \cdot b) \circ (1 \cdot b) = (a \cdot b) \circ 1$.

iii. By ii, JD3, i, and Lemma 1 i, $(a \circ 1) \cdot (b \circ 1) = ((a \circ 1) \cdot b) \circ 1 = ((a \cdot b) \circ (1 \cdot b)) \circ 1 = ((a \cdot b) \circ 1) \circ 1 = a \cdot b$.

iv. By JD3 and (ii), $a \cdot ((c \circ 1) \circ b) = (a \cdot (c \circ 1)) \circ (a \cdot b) = ((a \cdot c) \circ 1) \circ (a \cdot b)$. Also, $((c \circ 1) \circ b) \cdot a = ((c \circ 1) \cdot a) \circ (b \cdot a) = ((c \cdot a) \circ 1) \circ (b \cdot a)$.

In what follows, let X denotes a JDB-semigroup $(X, \circ, \cdot, 1)$ unless otherwise specified.

Definition 4. Let H be a nonempty subset of X. H is called a *sub JDB-semigroup* of X if H itself is a JDB-semigroup.

Remark 5. *Suppose X is a JDB-semigroup.*

i. If H is a sub JDB-semigroup of X, then $(H, \circ, 1)$ is a dual B-subalgebra of $(X, \circ, 1)$ and $1 \in H$.

ii. {1} and X are called trivial sub JDB-semigroups of X.

The following corollary shows for a subset to be a sub JDB-semigroup. This condition determines whether or not a nonempty subset of a JDB-semigroup is a sub JDB-semigroup.

The next Corollary follows from Theorem 1 and from the definition of the binary operator.

Corollary 1. (Sub JDB-semigroup Criterion) Let H be a nonempty subset of X. Then H is a sub JDB-semigroup of X if and only if $x \circ y$, $x \cdot y \in H$ for all $x, y \in H$.

Proof: Suppose H is a sub JDB-semigroup of X. Then H is a JDB-semigroup and so for all $x, y \in X$, $x \circ y$, $x \cdot y \in H$. Conversely, suppose $x \circ y$, $x \cdot y \in H$ for all $x, y \in H$. Then $(H, \circ, 1)$ is a dual B-subalgebra of $(X, \circ, 1)$ by Theorem 1. Since $x \cdot y \in H$, H is closed under $\cdot$. Since $H \subseteq X$, and $(X, \cdot)$ is a semigroup, then X is associative and so the operation $\cdot$ is left and right distributive over the operation $\circ$ follows.

Theorem 3. *Let X be a JDB-semigroup and $\{H_\alpha : \alpha \in \mathcal{I}\}$ be a nonempty collection of sub JDB-semigroup of X. Then $\bigcap_{\alpha \in \mathcal{I}} H_\alpha$ is also a sub JDB-semigroup of X.*

Proof: Since H_α is a sub JDB-semigroup for each α, then $1 \in H_\alpha$ for all $\alpha \in \mathcal{I}$. Hence, $1 \in \bigcap_{\alpha \in \mathcal{I}} H_\alpha$ and $\bigcap_{\alpha \in \mathcal{I}} H_\alpha \neq \varnothing$. Let $a, b \in \bigcap_{\alpha \in \mathcal{I}} H_\alpha$. Then $a, b \in H_\alpha$ for all $\alpha \in \mathcal{I}$. Since H_α is a sub JDB-semigroup of X for each α, $a \circ b$, $a \cdot b \in H_\alpha$ for all $\alpha \in \mathcal{I}$ by Corollary 1. It follows that $a \circ b$, $a \cdot b \in \bigcap_{\alpha \in \mathcal{I}} H_\alpha$. Hence, $\bigcap_{\alpha \in \mathcal{I}} H_\alpha$ is a sub JDB-semigroup of X.

Example 3. The set $H_1 = \{1, a\}$ in Example 1 is a sub JDB-semigroup, while the set $H_2 = \{1, a, b\}$ is not since $a \circ b = c \notin (H_2, \circ, 1)$ and $b \cdot b = c \notin (H_2, \cdot, 1)$.

Definition 5. A JDB-semigroup $(X, \circ, \cdot, 1)$ is called *commutative* if for all $a, b \in X$, $a \cdot b = b \cdot a$. Otherwise, it is called *noncommutative*.

Example 4. Consider the JDB-semigroup in Example 1, by routine calculation, $(X, \cdot)$ is commutative.

Note that not all JDB-semigroup is commutative as seen in the following example.

Example 5. Let $X = \{1, a, b, c\}$ be a set with the following tables:

$\circ$	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

$\cdot$	1	a	b	c
1	1	1	1	1
a	1	1	1	1
b	1	a	b	c
c	1	a	b	c

Then $b \cdot a = a \neq 1 = a \cdot b$ which implies that, $(X, \circ, \cdot, 1)$ is a noncommutative JDB-semigroup.

Definition 6. Let X be a JDB-semigroup. An element $b \in X$ is called *left unity* in X if $b \cdot a = a$ for all $a \in X$. An element $c \in X$ is called *right unity* in X if $a \cdot c = a$ for all $a \in X$. Moreover, an element $u \in X$ is called the *unity* in X if it is both left and right unity, that is, $a \cdot u = a = u \cdot a$ for all $a \in X$.

Example 6. Consider the JDB-semigroup in Example 5. The elements b and c are the left unity in X since for $b \in X$, $b \cdot a = a$, $b \cdot b = b$, $b \cdot c = c$ for all $a, b, c \in X$. Also, for $c \in X$, $c \cdot a = a$, $c \cdot b = b$, $c \cdot c = c$ for all $a, b, c \in X$.

Example 7. Consider the JDB-semigroup in Example 1, the element $a \in X$ is a right unity in X. In particular, $a \in X$ in Example 1 is also a left unity in X, hence a unity in X since $a \cdot a = a$, $b \cdot a = b = a \cdot b$, $c \cdot a = c = a \cdot c$ for all $a, b, c \in X$.

In view of Example 7, $1 \in X$ is not a unity in X.

This is illustrated in the next remark as immediate from Lemma 2(i).

Remark 6. *The element $1 \in X$ in a JDB-semigroup is not a unity in X when X is nontrivial.*

The unity in X, if it exist, is the identity element of X and is denoted by u.

The next theorem describes that $1 \in X$ is the unity if and only if X is the trivial JDB-semigroup $\{1\}$.

Theorem 4. *Let X be a JDB-semigroup. Then $1 \in X$ is a unity in X if and only if $X = \{1\}$.*

Proof: Suppose X is a JDB-semigroup and 1 is the unity in X. Assume on the contrary that there exist

$1 \neq a \in X$. Then $1 \cdot a = a = a \cdot 1$ since 1 is the unity. By Lemma 2(i), $1 \cdot b = 1 = b \cdot 1$ for all $b \in X$. Hence, $a = 1$, a contradiction. Hence $X = \{1\}$. Conversely, suppose $X = \{1\}$, then $1 \cdot 1 = 1$ which implies that 1 is the unity in X.

The following Theorem shows that the unity of the JDB-semigroup, if it exist, is unique.

Theorem 5. *Suppose X is a JDB-semigroup with unity. Then the unity in X is unique.*

Proof: Let X be a JDB-semigroup with unity $u \in X$. Then $u \cdot a = a = a \cdot u$ for all $a \in X$. Suppose $u' \in X$ is also a unity in X. Then $u' \cdot b = b = b \cdot u'$ for all $b \in X$. Now, $u = u \cdot u' = u'$. This implies that the unity in a JDB-semigroup is unique.

Definition 7. Let X be a JDB-semigroup with unity. An element a in a JDB-semigroup X is called a *unit* if and only if there exists $a' \in X$ such that $a \cdot a' = u = a' \cdot a$.

Example 8. Consider the JDB-semigroup X in Example 1. In view of Example 7, $a \in X$ is a unity in X. The elements $a, b, c \in X$ are units in X since for $a \in X$, there exist $a \in X$ such that $a \cdot a = a$, for $b \in X$, there exist $c \in X$ such that $b \cdot c = a = c \cdot b$ and for $c \in X$, there exist $b \in X$ such that $c \cdot b = a = b \cdot c$.

The next corollary follows from Theorem 4 and Definition 7.

Corollary 2. Suppose $X = \{1\}$ is the trivial JDB-semigroup. Then $1 \in X$ is both the unity and a unit in X.

Theorem 6. *Let X be a JDB-semigroup with unity and T be the set of all units in X. Then $a \cdot b \in T$ for all $a, b \in T$.*

Proof: Let $a, b \in T$. There exist $x, y \in X$ such that $a \cdot x = u = x \cdot a$ and $b \cdot y = u = y \cdot b$. Now $(a \cdot b) \cdot (y \cdot x) = a \cdot (b \cdot y) \cdot x = a \cdot u \cdot x = a \cdot x = u$. Similarly, $(y \cdot x) \cdot (a \cdot b) = u$. Hence, $(a \cdot b) \cdot (y \cdot x) = u = (y \cdot x) \cdot (a \cdot b)$. Thus, $a \cdot b$ is a unit and so $a \cdot b \in T$.

The next corollary follows from Theorem 6 and Corollary 1.

Corollary 3. Let X be a JDB-semigroup with unity and T be the set of all units in X. If $(T, \circ, 1)$ is a dual B-subalgebra, then T is a sub JDB-semigroup.

Definition 8. Let X be a nontrivial JDB-semigroup with unity. X is called a *JD-field* if the JDB-semigroup $(X, \cdot)$ is commutative and every element $a \in X$ is a unit.

Remark 7. *If X is a JD-field, X is a JDB-semigroup and $1 \in X$.*

Example 9. Consider the JDB-semigroup X in Example 1. In view of Example 8, X is a JD-field.

Definition 9. A sub JDB-semigroup F of X is called a *sub JD-field* of X if F is also JD-field.

Example 10. Consider the JDB-semigroup in Example 1. In view of Example 9. Let $F = \{1, a\}$ be a sub JDB-semigroup of X, then F is a sub JD-field of X.

Theorem 7. ***(Sub JD-field Criterion)*** *Let X be a JD-field. A nonempty subset $H \neq \{1\}$ of X is a sub JD-field if and only if*

i. $1 \in H$,

ii. $x \circ y$, $x \cdot y \in H$ for all $x, y \in H$,

iii. Every element $a \neq 1$ of H is a unit.

Proof: ($\Rightarrow$) (i) Suppose H is a sub JD-field of X. Since H is a sub JD-field, H is JD-field. In Remark 7, $1 \in H$. (ii) Since H is a sub JD-field, H is a sub JDB-semigroup, by Corollary 1, $x \circ y$, $x \cdot y \in H$ for all $x, y \in H$. (iii) Since H is a sub JD-field, H is a JD-field and every element $a \neq 1$ of H is a unit. ($\Leftarrow$) Conversely, suppose i, ii, iii holds. By Corollary 1, Definition 8, and Definition 9, H is a sub JDB-field.

Theorem 8. *Let X be a JD-field and $\{H_\alpha : \alpha \in \mathcal{F}\}$ be a nonempty collection of sub JD-fields in X. Then $\bigcap_{\alpha \in \mathcal{F}} H_\alpha$ is a sub JD-field of X.*

Proof: Let $\{H_\alpha : \alpha \in \mathcal{F}\}$ be a nonempty collection of sub JD-fields of X. By Theorem 7, $1 \in H_\alpha$ for all $\alpha \in \mathcal{F}$ which implies that $1 \in \bigcap_{\alpha \in \mathcal{F}} H_\alpha$. Suppose $x, y \in \bigcap_{\alpha \in \mathcal{F}} H_\alpha$. Then $x, y \in H_\alpha$ for all $\alpha \in \mathcal{F}$. Since H_α is a sub JD-field for all $\alpha \in \mathcal{F}$, then $x \circ y$, $x \cdot y \in H_\alpha$ for all $\alpha \in \mathcal{F}$. Hence, $x \circ y$, $x \cdot y \in \bigcap_{\alpha \in \mathcal{F}} H_\alpha$. Since H_α is a sub JD-field for all α, every element $a \neq 1$ of H_α is a unit, thus every element $a \in \bigcap_{\alpha \in \mathcal{F}} H_\alpha$ is a unit where $a \neq 1$. Thus, $\bigcap_{\alpha \in \mathcal{F}} H_\alpha$ is a sub JD-field X.

In [2], the authors introduced the notion of a normal subset of a dual B-algebra, a nonempty subset N of a dual B-algebra is said to be normal if for any $x \circ y, a \circ b \in N$, $(a \circ x) \circ (b \circ y) \in N$.

In what follows is a definition of JD-ideal which incorporates the definition of a normal dual B-algebra.

Definition 10. Let X be a JDB-semigroup. A subset F of X is called a *JD-ideal* of X if the following hold:

i. $1 \in F$,

ii. $(a \circ x) \circ (b \circ y) \in F$ for any $a \circ b$, $x \circ y \in F$,

iii. For any $a \in F$, $x \in X$ $a \cdot x, x \cdot a \in F$.

This means that the sub JDB-semigroup F in Definition 10(ii) is a normal subset of the dual B-algebra $(X, \circ, 1)$. The subsets $\{1\}$ and X are also JD-ideals of a JDB-semigroup X and are called *trivial* JD-*ideals* while other ideals are called *nontrivial JD-ideals*. In Example 5, the sets $F_1 = \{1, a\}$ and $F_2 = \{1, b\}$ are JD-ideals of X, while the set $F_3 = \{1, a, b\}$ is not since there exist $1 \circ a = a \in X$ and $1 \circ b = b \in X$ such that $(1 \circ 1) \circ (a \circ b) = 1 \circ c = c \notin F_3$. Consequently, F_3 is not a normal subset of X. Also, there exists $b \in F_3$ and $b \in X$ such $b \cdot b = c \notin F_3$.

Corollary 4. Let F be a JD-ideal of X. Then F is a sub JDB-semigroup of X.

Proof: Suppose F is a JD-ideal of X. Let $x, y \in F$. Since $1 \in F$ and $x = 1 \circ x$, $y = 1 \circ y \in F$, by DB2 and DB1 , $x \circ y = 1 \circ (x \circ y) = (1 \circ 1) \circ (x \circ y) \in F$, implies $x \circ y \in F$. Since F is a JD-ideal of X, then $x \in X$. By Definition 10 (iii), $x \cdot y$, $y \cdot x \in F$. This implies that every JD-ideal is a sub JDB-semigroup.

Theorem 9. *Suppose a sub JDB-semigroup contains F and $1 \in F$. Then F is a JD-ideal.*

Proof: Suppose S is a sub JDB-semigroup such that $F \subseteq S$ and $1 \in F$. It remains to show Definition 10(ii) and (iii). (ii) Suppose $a \circ b$, $x \circ y \in F$. Then $a \circ b$, $x \circ y \in S$. Since S is a sub JDB-semigroup, by Corollary 1, a, b, x, $y \in S$. Assume on the contrary that $(a \circ x) \circ (b \circ y) \notin F$, then $(a \circ x) \circ (b \circ y) \notin S$, a contradiction since S is a sub JDB-semigroup. (iii) Suppose $a \in F$, $x \in X$, then $a \in S$. Assume on the contrary that $a \cdot x$, $x \cdot a \notin F$, then $a \cdot x$, $x \cdot a \notin S$ which implies that $a \cdot x$, $x \cdot a \notin X$, a contradiction. Thus, F is a JD-ideal of a sub JDB-semigroup S of X.

Theorem 10. *Let X be a JDB-semigroup and $\{H_\alpha : \alpha \in \mathcal{I}\}$ be a nonempty collection of JD-ideals in X. Then $\bigcap_{\alpha \in \mathcal{I}} H_\alpha$ is a JD-ideal of X.*

Proof: Let $H = \bigcap_{\alpha \in \mathcal{I}} H_\alpha$. Note that $1 \in H_\alpha$ for all $\alpha \in \mathcal{I}$. Thus, $1 \in H$ and H is nonempty. Let $a \circ b$, $x \circ y \in H$. Then $a \circ b$, $x \circ y \in H_\alpha$ for all α. Since H_α is a JD-ideal for each α, it follows that $(a \circ x) \circ (b \circ y) \in H_\alpha$ for all α. Hence, $(a \circ x) \circ (b \circ y) \in H$. Let $a \in H$, $x \in X$. Since H_α is a JD-ideal for each α, then $a \cdot x$, $x \cdot a \in H$. Hence, H is a JD-ideal of X.

Remark 8. *The union of two JD-ideals is not necessarily a JD-ideal.*

This is illustrated in the next example.

Example 11. Consider the JDB-semigroup $(X, \circ, \cdot, 1)$ in Example 5. The set $F_1 = \{1, a\}$ and $F_2 = \{1, b\}$ are JD-ideals of X. Since $a \circ c = b \in F_1 \cup F_2$ but $c \notin F_1 \cup F_2$. Thus $F_1 \cup F_2 = \{1, a, b\}$ is not a JD ideal.

The following lemmas also hold in a JDB-semigroup and are necessary on the next theorem.

Lemma 3. *Let X be a dual B-algebra, then for all x, $y \in X$, $(x \circ y) \circ 1 = y \circ x$.*

Proof: Suppose x, $y \in X$ such that X is a dual B-algebra. By Lemma 1(ii) and (i), $(x \circ y) \circ 1 = y \circ [(x \circ 1) \circ 1] = y \circ x$.

Lemma 4. *Let F be the dual B-subalgebra of a dual B-algebra X. Let $a, b \in X$, if $a \circ b \in F$, then $b \circ a \in F$.*

Proof: Let $a \circ b \in F$. By Lemma 3, $b \circ a = (a \circ b) \circ 1$. Since $1 \in F$ and $a \circ b \in F$, then $(a \circ b) \circ 1 \in F$. Similarly, $a \circ b = (b \circ a) \circ 1 \in F$.

Theorem 11. *Suppose A be the sets of all subalgebras of a dual B-algebra X. Let $N \in A$. Then the following are equivalent.*

(i) N is a normal dual B-subalgebra;

(ii) If $x \in X$, $y \in N$, then $(y \circ x) \circ x \in N$.

Proof: (i) $\Rightarrow$ (ii): Let $x \in X$, $y \in N$. Since N is a dual B-subalgebra, $y \circ 1 \in N$ and $x \circ x = 1 \in N$. Since N is normal, by DB2, $(y \circ x) \circ x = (y \circ x) \circ (1 \circ x)$.
(ii) $\Rightarrow$ (i): Let $x \circ y$, $a \circ b \in N$. By Lemma 4, $b \circ a \in N$. By Theorem 2 and (ii), $(b \circ 1) \circ (a \circ 1) = ((a \circ b) \circ (a \circ 1)) \circ (a \circ 1) \in N$. By applying DB3 twice, $(b \circ x) \circ (a \circ x) = ((a \circ 1) \circ (b \circ x)) \circ x = (((b \circ 1) \circ (a \circ 1)) \circ x) \circ x \in N$. Thus, $(b \circ x) \circ (a \circ x) \in N$. Since N is a dual B-subalgebra, $((b \circ x) \circ (a \circ x)) \circ (x \circ y) \in N$. By Lemma 1(ii), Lemma 3, and Theorem 2, $((b \circ x) \circ (a \circ x)) \circ (x \circ y) = (a \circ x) \circ (((b \circ x) \circ 1) \circ (x \circ y)) = (a \circ x) \circ ((x \circ b) \circ (x \circ y)) = (a \circ x) \circ (b \circ y)$. Thus, $(a \circ x) \circ (b \circ y) \in N$ and therefore N is normal.

Proposition 1. *Let S be a sub JDB-semigroup of X. Then S is a normal dual B-subalgebra of X if and only if S is a JD-ideal of X.*

Proof: ($\Rightarrow$) Suppose S is a normal dual B-algebra of X. Let a, b, x, $y \in S$. By Remark 5(i), $1 \in S$. Since S is normal, $(a \circ x) \circ (b \circ y) \in S$ for any $a \circ b$, $x \circ y \in S$. Since S is a sub JDB-semigroup, then for any $a \in S$, $x \in X$, $a \cdot x$, $x \cdot a \in S$. ($\Leftarrow$) Now, suppose S is a JD-ideal of X. Then S is a normal subset of X. By Corollary 4 and Remark 5(i), S is a normal dual B-subalgebra of X.

Definition 11. Let a, $b \in X$. The subset $Z(X)$ of X is called the *center* of X if $Z(X) = \{a \in X | a \cdot b = b \cdot a$ for all $b \in X\}$.

Example 12. Consider the JDB-semigroup in Example 1. The JDB-semigroup $Z(X) = X$ is the center of X.

Remark 9. Let X be a JDB-semigroup.

(i) By Lemma 2(i) it follows that 1 is in $Z(X)$, consequently $Z(X)$ is nonempty.

(ii) If the JDB-semigroup is commutative, then $Z(X) = X$. Moreover, the center of every JD-field is itself.

Theorem 12. Let X be a JDB-semigroup. Then $Z(X)$ is a sub JDB-semigroup of X.

Proof: Let $a, b \in Z(X)$ and $x \in X$. By JD3, $x \cdot (a \circ b) = (x \cdot a) \circ (x \cdot b) = (a \cdot x) \circ (b \cdot x) = (a \circ b) \cdot x$. Hence, $a \circ b \in Z(X)$. Furthermore, $x \cdot (a \cdot b) = (x \cdot a) \cdot b = (a \cdot x) \cdot b = a \cdot (x \cdot b) = a \cdot (b \cdot x) = (a \cdot b) \cdot x$, so $a \cdot b \in Z(X)$. Thus, $Z(X)$ is a sub JDB-semigroup of X.

IV. Conclusion

This research introduced and investigated the JDB-semigroup; the findings of this study proves the existence of semigroup to dual B-algebra. This study also describes the relationship between JDB-semigroup and JB-semigroup. Some properties of the JDB-semigroup such as those that involve its elements, and the intersection of sub JDB-semigroups, sub JD-fields, and sub JD-ideals are also sub JDB-semigroup, sub JD-field, and sub JD-ideal, respectively. In addition, the characterizations of sub JDB-semigroup and sub JD-field are provided as the sub JDB-semigroup criterion and sub JD-field criterion, respectively. Future research on the homomorphism of the JDB-semigroup and investigation of its isomorphism theorems would be interesting to study.

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Contribution of individual authors to the creation of a scientific article (ghostwriting policy)

Katrina Belleza Fuentes proposed the study's methodology, including the extraction of ideas and verification of the results.
Joshue G. Derecho for constructing the study's findings.

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Conflicts of Interest

This is to confirm that there is no conflict of interest between any of the authors.