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Paraconsistent da Costa Weakening of Intuitionistic Negation: What does it mean?

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Abstract—**In this paper we consider the systems of weakening of intuitionistic negation logic mZ, introduced in [1], [2], which are developed in the spirit of da Costa's approach. We take a particular attention on the philosophical considerations of the paraconsistent mZ logic w.r.t. the constructive semantics of the intuitionistic logic, and we show that mZ is a subintuitionistic logic. Hence, we present the relationship between intuitionistic and paraconsistent subintuitionistic negation used in mZ.**
Then we present a significant number of examples for this subintuitionistic and paraconsistent mZ logics: Logic Programming with Fiting's fixpoint semantics for paraconsistent weakening of 3-valued Kleene's and 4-valued Belnap's logics. Moreover, we provide a canonical construction of infinitary-valued mZ logics and, in particular, the paraconsistent weakening of standard Zadeh's fuzzy logic and of the Gödel-Dummet t-norm intermediate logics.

Index Terms—**Da Costa paraconsistent logic, Intuitionistic logic, Majkić's systems mZ.**

I. Introduction

In what follows we will try to summarize, in a short introduction, the previous historic approach to two important concepts in the logics: paraconsistency and constructivism.

A *paraconsistent* logic is a logical system that attempts to deal with contradictions in a discriminating way. Alternatively, paraconsistent logic is the subfield of logic that is concerned with studying and developing paraconsistent (or inconsistency-tolerant) systems of logic. Paraconsistent logics are propositionally weaker than classical logic; that is, they deem fewer propositional inferences valid. The point is that a paraconsistent logic can never be a propositional extension of classical logic, that is, propositionally validate everything that classical logic does. In that sense, then, paraconsistent logic is more conservative or cautious than classical logic.

The interpretation of negation is different in intuitionist logic than in classical logic. In classical logic, the negation of a statement asserts that the statement is false; to an intuitionist, it means the statement is refutable (e.g., that there is a counterexample). There is thus an asymmetry between a positive and negative statement in intuitionism. If a statement A is provable, then it is certainly impossible to prove that there is no proof of A. But even if it can be shown that no disproof of A is possible, we cannot conclude from this absence that there is a proof of A. Thus A is a stronger statement than $\neg\neg A$.

Intuitionistic logic allows $A \vee \neg A$ not to be equivalent to true, while paraconsistent logic allows $A \wedge \neg A$ not to be equivalent to false. Thus it seems natural to regard paraconsistent logic as the *dual* of intuitionistic logic. However, intuitionistic logic is a specific logical system whereas paraconsistent logic encompasses a large class of systems. Accordingly to this historic approach, the dual notion to paraconsistency is called paracompleteness, and the dual of intuitionistic logic (a specific paracomplete logic) is a specific paraconsistent system called anti-intuitionistic or dual-intuitionistic logic (sometimes referred to as Brazilian logic, for historical reasons) [3], [4], [5], [6]. The duality between the two systems is best seen within a sequent calculus framework.

The intuitionistic implication operator cannot be treated like "$(\neg_C A \vee B)$", but as a modal formula $\Box(\neg_C A \vee B)$ where $\neg_C$ is the classic negation and $\Box$ universal modal S4 operator with reflexive and transitive accessibility relation R in Kripke semantics. Dual-intuitionistic logic [7] contains a connective $\multimap$ known as pseudo-difference which is the dual of intuitionistic implication. Very loosely, $A \multimap B$ can be read as "A but not B" and is equivalent to a modal formula $\Diamond(A \wedge \neg_C B)$, where $\Diamond$ is the existential modal operator with inverse accessibility relation R^{-1} in a Kripke-like semantics [8].

Dual of intuitionistic logic has been investigated to varying degrees of success using algebraic, relational, axiomatic and sequent-based perspectives. The concept of anti-intuitionism, proposed through the concept of a dual intuitionistic logic, was already mentioned in the forties by K.Popper (without any formalism), but he disapproved such a logic as "too weak to be useless". In fact K.Popper puts in the Logic of Scientific Discovery [9]:
"The falsifying mode of inference here referred to - the way in which the falsification of conclusion entails the falsification of the system from which it is derived - is the modus tolens of classic logic."

Consequently, dual intuitionistic logic can be labeled as "falsification logic". In the falsification logic truth is essentially non constructive as opposed to falsity that is conceived constructively. In intuitionistic logic, instead, falsity is essentially non constructive as opposed to truth that is conceived constructively.

Thus, historically, the main research about the relationships between intuitionistic and paraconsistent logic was directed toward an exploration of their dual and opposite properties instead of investigation of their possible common properties. Such common properties are interesting in order to obtain the logics that may have a reasonable balance of both opposite (dual) properties: the logics where both, $\neg(A \wedge \neg B)$ is not false and $\neg(A \vee \neg B)$ is not true.

This consideration was the basic motivation at the beginning of my investigation of intuitionism and paraconsistency, in order to obtain an useful logic for significant practical applications as well. In fact, after a meeting with Walter Carnielli, M.Coniglio and J.Y.Béziau at IICAI-2007 (Pune, India), I decided to dedicate much more time to these problems. My first result was the publication of an autoreferential semantics for modal logics based on a complete distributive lattice [10]. In this research I dedicated the last section exclusively to the paraconsistency of this new semantics w.r.t. the formal LFI system [6] which I obtained in preprint personally from Walter.

My decisive advances in this direction of research was published in [1] and then upgraded in [2] by considering the paraconsistent properties of mZ_n logics. Consequently, in this paper, we will consider in a more detailed way, the dual quasi-intuitionistic properties of this mZ_n logic.

There are different approaches to paraconsistent logics. The first one is the non constructive approach, based on abstract logic (as LFI [6]), where logic connectives and their particular semantics are not considered. The second one is the constructive approach and is divided in two parts: an axiomatic proof theoretic (in da Costa [11] and [12], [13], [14]), and a many-valued model theoretic [10] based on truth-functional valuations (that is, it satisfies the truth-compositionality principle). The best scenario is when we obtain both, the proof and the model theoretic definition, which are mutually sound and complete.

One of the main founders with Stanislav Jaskowski [15], da Costa, built his propositional paraconsistent system C_ω in [11] by weakening the logic negation operator $\neg$, in order to avoid the explosive inconsistency [6], [16] of the classic propositional logic, where the ex falso quodlibet proof rule $\frac{A,\ \neg A}{B}$ is valid. In fact, in order to avoid this classic logic rule, he changed the semantics for the negation operator, so that:

- NdC1: in these calculi the principle of non-contradiction, in the form $\neg(A\wedge\neg A)$, should not be a generally valid schema, but if it does hold for formula A, it is a well-behaved formula and is denoted by A°;
- NdC2: from two contradictory formulae, A and $\neg A$, it would not in general be possible to deduce an arbitrary formula B. That is, it does not hold the falso quodlibet proof rule $\frac{A,\ \neg A}{B}$;
- NdC3: it should be simple to extend these calculi to corresponding predicate calculi (with or without equality);
- NdC4: they should contain most parts of the schemata and rules of classical propositional calculus which do not infer with the first conditions

In fact, Da Costa's paraconsistent propositional logic is made up of the unique Modus Ponens inferential rule (MP), $A, A\Rightarrow B \vdash B$, and two axiom subsets. But before stating them we need the following definition as it is done in da Costa's systems (c.f. [11, p.500]), which uses three binary connectives, $\wedge$ for conjunction, $\vee$ for disjunction and $\Rightarrow$ for implication:

Definition 1: Let A be a formula and $1\leq n<\omega$. Then, we define $A^\circ, A^{\mathrm{n}}, A^{(\mathrm{n})}$ as follows:

$A^\circ =_{\text{def}} \neg(A\wedge\neg A)$, $\quad A^{\mathrm{n}} =_{\text{def}} A\overbrace{^{\circ\circ\cdots\circ}}^{n}$, and

$A^{(\mathrm{n})} =_{\text{def}} A^1\wedge A^2\wedge\cdots\wedge A^{\mathrm{n}}$.

The first one is for the positive propositional logic (without negation), composed by the following eight axioms, borrowed from the classic propositional logic of the Kleene L_4 system, and also from the more general propositional *intuitionistic* system (these two systems differ only regarding axioms with the negation operator),

(IPC^+) POSITIVE LOGIC AXIOMS:
(1) $A\Rightarrow(B\Rightarrow A)$
(2) $(A\Rightarrow B)\Rightarrow((A\Rightarrow(B\Rightarrow C))\Rightarrow(A\Rightarrow C))$
(3) $A\Rightarrow(B\Rightarrow(A\wedge B))$
(4) $(A\wedge B)\Rightarrow A$
(5) $(A\wedge B)\Rightarrow B$
(6) $A\Rightarrow(A\vee B)$
(7) $B\Rightarrow(A\vee B)$
(8) $(A\Rightarrow C)\Rightarrow((B\Rightarrow C)\Rightarrow((A\vee B)\Rightarrow C))$

We change the original axioms for negation operator of the classic propositional logic, by defining the semantics of negation operator by the following subset of axioms:

(NLA) LOGIC AXIOMS FOR NEGATION:
(9) $A\vee\neg A$
(10) $\neg\neg A\Rightarrow A$
(11) $B^{(n)}\Rightarrow((A\Rightarrow B)\Rightarrow((A\Rightarrow\neg B)\Rightarrow\neg A))$ (Reductio relativization axiom)
(12) $(A^{(n)}\wedge B^{(n)})\Rightarrow((A\wedge B)^{(n)}\wedge(A\vee B)^{(n)}\wedge(A\Rightarrow B)^{(n)})$
□

It is easy to see that the axiom (11) relativizes the classic *reductio* axiom $(A\Rightarrow B)\Rightarrow((A\Rightarrow\neg B)\Rightarrow\neg A)$ (which is equivalent to the contraposition axiom

$$(A\Rightarrow\neg B)\Rightarrow(B\Rightarrow\neg A)$$

and the trivialization axiom $\neg(A\Rightarrow A)\Rightarrow B$), *only* for propositions B such that $B^{(n)}$ is valid, and in this way avoids the validity of the classic ex falso quodlibet proof rule. It provides a qualified form of reduction, helping to prevent general validity of $B^{(n)}$ in the paraconsistent logic C_n. The axiom (12) regulates only the propagation of n-consistency. It is easy to verify that n-consistency also propagates through negation, that is, $A^{(n)}\Rightarrow(\neg A)^{(n)}$ is provable in C_n. So that for any fixed n (from 0 to ω) we obtain a particular da Costa paraconsistent logic C_n.

It is well known that the classic propositional logic based on the classic 2-valued complete distributive lattice $(\mathbf{2},\leq)$ with the set $\mathbf{2}=\{0,1\}$ of truth values, has a truth-compositional model theoretic semantics. For this da Costa calculi is not given any truth-compositional model theoretic semantics instead.

Based on these observations, in [1] are explained some weak properties of Da Costa weakening for a negation operator and it was shown that negation is not antitonic, differently from the negations in the classic and intuitionistic propositional logics (that have the truth-compositional model theoretic semantics).

The negation in the classic and intuitionistic logics are not paraconsistent (see for example Proposition 30, pp 118, in [10]), so that basic idea in [1] was to make a weakening of the intuitionistic negation by considering only its general antitonic property. In fact, the formula $(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$ is a thesis in both classic and intuitionistic logics. Consequently, our idea was to use da Costa weakening of the *intuitionistic* negation [1], [2], that is, to define the system mZ_n for each n by adding the following axioms to the system IPC^+:

(9b) $(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$
(10b) $1 \Rightarrow \neg 0$
(11b) $A \Rightarrow 1,\ 0 \Rightarrow A$
(12b) $(\neg A \wedge \neg B) \Rightarrow \neg(A \vee B)$

After that was demonstrated in [2] that the axioms (11) and (12) are redundant in mZ_n, in the sense that they can be proved by another axioms. Moreover, the propagation axioms
$A^{(\mathrm{n})} \Rightarrow (\neg A)^{(\mathrm{n})}$
can be fully proved in systems mZ_n and mCZ_n(obtained by adding the axiom
(13b) $\neg(A \wedge B) \Rightarrow (\neg A \vee \neg B)$).

Remark: From the fact that mZ_n logic satisfy da Costa axioms for all $n \geq 1$, in what follows we will write simply mZ logic.
□

The plan of this paper is as follows:
In Section 2 we present the semantics of modal additive negations based on the Birkhoff's polarity, that is, so called split negations, and demonstrate that these negations satisfy the da Costa's paraconsistent axiom system. In Section 3 we analyze the principle of constructive negation and we show that the a paraconsistent split negation, used in mZ system, is constructive as intuitionistic negation, which is a special case of a split (but non paraconsistent) negation. Then, we present a paraconsistent Heyting algebras and show that they do not satisfy the da Costa's non-contradiction principle.

The main theoretical and philosophical results are presented in Section 4 by demonstration that mZ system is a *subintuitionistic* constructive paraconsistent logic and is provided the relationship between intuitionistic and this new paraconsistent negation in mZ. The rest of this paper is dedicated to significant applications of the subintuitionistic paraconsistent mZ logics. Section 5 is dedicated to Logic Programming with Fitting's fixpoint semantics (w.r.t. the knowledge ordering) by using mZ logics with minimal cardinalities, obtained by paraconsistent da Costa's weakening of the 3-valued Kleene and 4-valued bilattice-based Belnap's logics. Finally, in Section 6 we present the canonical constructions of infinite-valued mZ logics. In particular, we present the paraconsistent weakening of the classic simplest (Zadeh) fuzzy logic and of the Gödel-Dummet t-norm intermediate logic.

II. Semantics of negation based on Bikhoff's polarity

It was demonstrated in [1] (Proposition 3) that the positive fragment of mZ system corresponds to the distributive lattice $(X, \leq)$ (the positive fragment of the Heyting algebra), where the logic implication corresponds to the relative pseudocomplement and 0 and 1 are the bottom and top elements in X respectively.

Now we may introduce a hierarchy of negation operators [10] for many-valued logics based on complete lattices of truth values $(X, \leq)$, w.r.t their homomorphic properties. The negation with the lowest requirements (antitonic) denominated "general" negation can be defined in any complete lattice (for example, see [1]):

Definition 2: Hierarchy of Negation operators: Let $(X, \leq, \wedge, \vee)$ be a complete lattice. Then we define the following hierarchy of negation operators on it:

1. A `general` negation is a monotone mapping between posets ($\leq^{OP}$ is inverse of $\leq$),
$\neg : (X, \leq) \rightarrow (X, \leq)^{OP}$, such that $\{1\} \subseteq \{y = \neg x \mid x \in X\}$.
2. A `split` negation is a general negation extended into join-semilattice homomorphism,
$\neg : (X, \leq, \vee, 0) \rightarrow (X, \leq, \vee, 0)^{OP}$,
with $(X, \leq, \vee, 0)^{OP} = (X, \leq^{OP}, \vee^{OP}, 0^{OP})$, $\vee^{OP} = \wedge$, $0^{OP} = 1$.
3. A `constructive` negation is a general negation extended into full lattice homomorphism,
$\neg : (X, \leq, \wedge, \vee) \rightarrow (X, \leq, \wedge, \vee)^{OP}$,
with $(X, \leq, \wedge, \vee)^{OP} = (X, \leq^{OP}, \wedge^{OP}, \vee^{OP})$, and $\wedge^{OP} = \vee$.
4. A `De Morgan` negation is a constructive negation if the lattice homomorphism is an involution ($\neg\neg x = x$).
□

The names given to these different kinds of negations follow from the fact that: a split negation introduces the second right adjoint negation, a constructive negation satisfies the constructive requirement (as in Heyting algebras) $\neg\neg x \geq x$, while De Morgan negation satisfies the well known De Morgan laws, as follows:

Lemma 1: Negation properties): Let $(X, \leq)$ be a complete lattice. Then the following properties for negation operators hold: for any $x, y \in X$,

1. for general negation:
$\neg(x \vee y) \leq \neg x \wedge \neg y, \quad \neg(x \wedge y) \geq \neg x \vee \neg y,$
with $\neg 0 = 1$.
2. for split negation: $\neg(x \vee y) = \neg x \wedge \neg y$, $\neg(x \wedge y) \geq \neg x \vee \neg y$. It is an additive modal operator with right adjoint (multiplicative) negation $\sim: (X, \leq)^{OP} \rightarrow (X, \leq)$, and Galois connection $\neg x \leq^{OP} y$ iff $x \leq \sim y$, such that $x \leq \sim \neg x$ and $x \leq \neg \sim x$.
3. for constructive negation: $\neg(x \vee y) = \neg x \wedge \neg y$, $\neg(x \wedge y) = \neg x \vee \neg y$. It is a selfadjoint operator, $\neg = \sim$, with $x \leq \neg\neg x$ satisfying `proto` De Morgan inequalities
$\neg(\neg x \vee \neg y) \geq x \wedge y$ and $\neg(\neg x \wedge \neg y) \geq x \vee y$.
4. for De Morgan negation ($\neg\neg x = x$): it satisfies also De Morgan laws $\neg(\neg x \vee \neg y) = x \wedge y$ and $\neg(\neg x \wedge \neg y) = x \vee y$, and is contrapositive, i.e., $x \leq y$ iff $\neg x \geq \neg y$.

(Proof can be found in [10]).

Remark: We can see (as demonstrated in [1]) that the negation in the system mZ without axiom (12b) is a particular case of *general* negation, that the negation in the whole system mZ is a *split* negation, while the negation in the system mCZ [1] is a *constructive* negation.

The Galois connections can be obtained from any binary relation based on a set $\mathcal{W}$ [17] (Birkhoff *polarity*) in a canonical way: If $(\mathcal{W},\mathcal{R})$ is a set with a particular relation based on a set $\mathcal{W}$, $\mathcal{R}\subseteq\mathcal{W}\times\mathcal{W}$, with mappings $\lambda:\mathcal{P}(\mathcal{W})\rightarrow\mathcal{P}(\mathcal{W})^{OP}$, $\varrho:\mathcal{P}(\mathcal{W})^{OP}\rightarrow\mathcal{P}(\mathcal{W})$, where $\mathcal{P}$ is the powerset operation, such that for subsets $U,V\in\mathcal{P}(\mathcal{W})$,

$\lambda U=\{w\in\mathcal{W}\mid\forall u\in U.((u,w)\in\mathcal{R})\}$,

$\rho V=\{w\in\mathcal{W}\mid\forall v\in V.((w,v)\in\mathcal{R})\}$,

where $(\mathcal{P}(\mathcal{W}),\subseteq)$ is the *powerset poset* complete distributive lattice with bottom element empty set $\emptyset$ and top element $\mathcal{W}$, and $\mathcal{P}(\mathcal{W})^{OP}$ its dual (with $\subseteq^{OP}$ inverse of $\subseteq$), then we have the induced Galois connection $\lambda\dashv\rho$, i.e., $\lambda U\subseteq^{OP}V$ iff $U\subseteq\rho V$.

It is easy to verify that λ and ρ are two antitonic set-based operators which invert empty set $\emptyset$ into $\mathcal{W}$, thus can be used as two set-based negation operators. The negation as a modal operator has a long history [18].

Let us consider a case of complete distributive lattices used in Kripke semantics for the intuitionistic propositional logic:

Definition 3: Let $(\mathcal{W},\sqsubseteq)$ be a poset. A subset $S\subseteq\mathcal{W}$ is said to be hereditary if $x\in S$ and $x\sqsubseteq x'$ implies $x'\in S$. We denote the subset of all hereditary subsets of $\mathcal{P}(\mathcal{W})$ by $\mathcal{H}_W$ so that $(\mathcal{H}_W,\subseteq,\bigcap,\bigcup)$ is a sublattice of the powerset lattice $(\mathcal{P}(\mathcal{W}),\subseteq,\bigcap,\bigcup)$, with bottom element (empty set) $\emptyset$ and top element $\mathcal{W}$ in $\mathcal{H}_W$ respectively.

We define also the algebraic implication operator $\rightharpoonup$ by the relative pseudocomplement for sets, given by

$S\rightharpoonup S'=\bigcup\{Z\in\mathcal{H}_W\mid Z\bigcap S\subseteq S'\}$.

□

The hereditary sets in $\mathcal{H}_W$ are closed under set intersection and union, thus it is closed also under a relative pseudo-complement operator $\rightharpoonup$ which is expressed by using set union and intersection. As a result, we obtain the positive fragment of Heyting algebra $(\mathcal{H}_W,\subseteq,\bigcap,\bigcup,\rightharpoonup)$.

Let $\mathfrak{R}$ be the class of such binary incompatibility relations $\mathcal{R}\subseteq\mathcal{W}\times\mathcal{W}$ which are also `hereditary`, i.e.,

if $(u,w)\in\mathcal{R}$ and $(u,w)\preceq(u',w')$ then $(u',w')\in\mathcal{R}$, where

$(u,w)\preceq(u',w')$ iff $u'\sqsubseteq u$ and $w\sqsubseteq w'$,

so that $\sqsubseteq\circ\mathcal{R}\circ\sqsubseteq\;\subseteq\;\mathcal{R}$, where $\circ$ is a composition of binary relations.

In this case of the Birkhof polarity for any $U\in\mathcal{H}_W$ we obtain [1] that $\lambda U=\{w\in\mathcal{W}\mid\forall u\in U.((u,w)\in\mathcal{R})\}\in\mathcal{H}_W$, that is, $\mathcal{H}_W$ is closed under λ as well. Consequently, we obtain an extended positive fragment of Heyting algebra with this antitonic negation operation λ, i.e., the algebra on hereditary subsets $(\mathcal{H}_W,\subseteq,\bigcap,\bigcup,\rightharpoonup,\lambda)$.

Analogously to demonstration given in [1], it is easy to see that, for any given hereditary incompatibility relation $\mathcal{R}$, the additive algebraic operator λ can be used as a split negation for mZ logic.

Corollary 1: [2] *Each split negation, based on the hereditary incompatible relation of Birkhoff polarity, satisfies the Da Costa weakening axioms (11) and (12).*

It was shown [2] that this system mZ satisfies the Da Costa's requirements, NdC1,NdC2, NdC3 and NdC4 as well, because the positive fragment of this logics is equal to the positive fragment of propositional logic, so that it is a conservative extension of the positive propositional logic.

The mZ is many-valued propositional logics with a set $\mathcal{B}$ of logic truth values, defined as follows:

Definition 4: The set of logic truth values $\mathcal{B}$ is defined by the (order preserving) isomorphism

$$is:(\mathcal{H}_W,\subseteq,\bigcap,\bigcup,\rightharpoonup,\lambda)\rightarrow(\mathcal{B},\leq,\wedge,\vee,\Rightarrow,\neg)\qquad(1)$$

such that $is(\emptyset)=0$ and $is(\mathcal{W})=1$. Here $\Rightarrow$ is the relative-pseudo complement (as in any Heyting algebra for the intuitionistic implication) in the distributive complete lattice $\mathcal{B}$ and $\neg=is\circ\lambda\circ is^{-1}:\mathcal{B}\longrightarrow\mathcal{B}$.

□

Differently from the original da Costa weakening of the classical negation that results in a non truth-functional logics, mZ is a system of many-valued truth-functional logics.

III. The intuitionistic interpretation of negation and logic constructivism

The fundamental characteristic of the IPC is that negation is a derived operation from the constructive implication, that is $\neg_I A$ is defined by $A\Rightarrow 0$. Thus, in order to analyze the intuitionistic negation $\neg_I$ we have to consider the intuitionistic implication $\Rightarrow$.

In Kolmogorov [19] he took care to lay down the meaning of constructive implication:

"*The meaning of the symbol $A\Rightarrow B$ is exhausted by the fact that, once convinced of the truth of A, we have to accept the truth of B too.*"

From this point of view the ex falso (quodlibet) principle (EF) $\neg A\Rightarrow(A\Rightarrow B)$ was questionable, to say the least. In fact he decided to reject it (in the sense of paraconsistent logic). He argued that axiom now considered does not have and cannot have any intuitive foundation, since it asserts something about the consequence of something impossible. in 1932 Kolmogorov published [20] a full version of this 'problematic interpretation', by considering a proposition A as a problems and constructive truth of A comes to "we have a solution of A". Thus, a problem (proposition) $A\Rightarrow B$ is formulated as "given a solution of A, find solution of B".

The ex falso axiom $\neg_I A\Rightarrow(A\Rightarrow B)$ was accepted by Kolmogorov on the strength of the following *convention*:

"*As far as problem $\neg_I A\Rightarrow(A\Rightarrow B)$ is concerned, as soon as $\neg_I A$ is solved, the solution of A is impossible, and the problem $A\Rightarrow B$ has no content. In what follows, the proof that a problem is without content will always be considered as its solution.*"

Thus this *convention* about negation is highly non constructive. The modern formulation of intuitionistic implication appears first time in 1934 in Heyting [21], p.14:

"*A proof of a proposition consists of the realization of the construction demanded by it. $A\Rightarrow B$ means the intension on a construction, which leads from* any *proof of A to a proof of B.*"

Kolmogorov never returned to intuitionistic logic and the matter of EF rule, Heyting on the other hand returned to this principle in his intuitionism where he recognized that in the case of a false antecedent the construction interpretation is problematic:

"*Now suppose that $\neg_I A$ is true, that is, we have deduced a contradiction from supposition that A were carried out. Then, in sense, this can be considered as a construction, which, joined to a prof of A (which can not exist) leads to a proof of B. I shell interpret then implication in this wider sense.*"

Thus Heyting' justification of the intuitionistic negation is, albeit hesitant, the standard argument of today.

We recall that the current intuitionistic meaning of logic connectives are as follows:

- a proof of $A \wedge B$ consists of a proof of A and a proof of B plus the conclusion $A \wedge B$, or in Kripke semantics, for any possible world $x \in \mathcal{W}$,
 $\mathcal{M} \models_x A \wedge B$ iff $\mathcal{M} \models_x A$ and $\mathcal{M} \models_x B$.
- a proof of $A \vee B$ consists of a proof of A or a proof of B plus the conclusion $A \vee B$, or in Kripke semantics, for any possible world $x \in \mathcal{W}$,
 $\mathcal{M} \models_x A \vee B$ iff $\mathcal{M} \models_x A$ or $\mathcal{M} \models_x B$.
- a proof of $A \Rightarrow B$ consists of a method (or algorithm) of converting any proof of A into a proof of B, or in Kripke semantics, for any possible world $x \in \mathcal{W}$,
 $\mathcal{M} \models_x A \Rightarrow B$ iff $(\forall y \in \mathcal{W})((x,y) \in R_\Box$ implies $(\mathcal{M} \models_y A$ implies $\mathcal{M} \models_y B))$,
 where $R_\Box$ is a reflexive and transitive relation, so that $A \Rightarrow B$ is equivalent to a modal formula $\Box(A \Rightarrow_C B) \equiv \Box(\neg_C A \vee B)$, where $\Box$ is the universal modal operator "necessary" of S4 modal logic and $\Rightarrow_C, \neg_C$ the classical implication and negation respectively.
 Notice that $\neg_I A$ is obtained when B is the falsum 0, so that $\neg_I A \equiv \Box \neg_C A$.
- no proof of 0 (falsum) exists, i.e., for each $x \in \mathcal{W}$, $\mathcal{M} \not\models_x 0$, that is, "not $\mathcal{M} \models_x 0$".

These considerations give more justifications in our attempt to modify this intuitionistic unconstructive negation into another more constructive and paraconsistent as well.

Proposition 1: A paraconsistent negation in mZ logic is paraconsistently-constructive, that is, $\neg = \Box_P \neg_c$ where $\neg_c$ is the classical negation and $\Box_P$ is new universal paraconsistent modal operator.

Proof: Let us consider the Kripke semantics of the intuitionistic negation $\neg_I A$ (equivalent to $A \Rightarrow 0$):

$\mathcal{M} \models_x \neg_I A$ iff $\mathcal{M} \models_x A \Rightarrow 0$
iff $\forall y((x,y) \in R_\Box$ implies $(\mathcal{M} \models_y A$ implies $\mathcal{M} \models_y 0)$
iff $\forall y((x,y) \in R_\Box$ implies (not $\mathcal{M} \models_y A$ or $\mathcal{M} \models_y 0)$
iff $\forall y((x,y) \in R_\Box$ implies not $\mathcal{M} \models_y A)$
iff $\mathcal{M} \models_x \Box \neg_C A$,

where $\neg_C$ is the classic negation and $\Box$ is the universal modal "necessity" (S4) operator, with the reflexive+transitive accessibility relation $R_\Box$ between theoretical constructions (from the Brower's constructive point of view). So that $(x,y) \in R_\Box$ means that a theoretical construction (proof) y is a result of positive development of a theoretical construction (proof) x.

Thus, from this constructive point of view, an informal Kripke semantics is as follows:

"*$\neg_I A$ is proved in the framework x iff in the framework of every possible construction y (which is the result of some development of the construction x) A is not proved.*"

The Kripike semantics for a paraconsistent negation $\neg$ in mZ logic, based on an incompatibility relation $\mathcal{R}$ (from Birkhoff's polarity), is defined by:

$\|\neg A\| = \lambda(\|A\|) = \{x \in \mathcal{W} \mid \forall y \in \|A\|.(y,x) \in \mathcal{R}\}$, i.e.,
$\mathcal{M} \models_x \neg A$ iff $\forall y(\mathcal{M} \models_y A$ implies $(y,x) \in \mathcal{R})$
iff $\forall y($not$\mathcal{M} \models_y A$ or $(y,x) \in \mathcal{R})$
iff $\forall y((y,x) \notin \mathcal{R}$ implies not $\mathcal{M} \models_y A)$
iff $\forall y((x,y) \in R_{\Box_P}$ implies not $\mathcal{M} \models_y A)$
iff $\mathcal{M} \models_x \Box_P \neg_C A$,

where $R_{\Box_P} = ((\mathcal{W} \times \mathcal{W}) \backslash \mathcal{R})^{-1}$ is the paraconsistently-constructive accessibility relation for the universal *paraconsistent* modal operator $\Box_P$.

$\Box$

What we obtain for the mZ logic is that it is a bimodal logic with two universal modal operators, the necessity universal modal operator $\Box$ (with accessibility binary relation equal to a poset $(\mathcal{W}, \sqsubseteq)$) used for the intuitionistic implication $\Rightarrow$ equal to $\Box \Rightarrow_C$ where $\Rightarrow_C$ is standard (classical) implication, and the universal paraconsistent modal operator $\Box_P$ (with the accessibility binary relation $R_{\Box_P} = ((\mathcal{W} \times \mathcal{W}) \backslash \mathcal{R})^{-1}$ derived from the *hereditary* incompatibility relation $\mathcal{R} \in \mathfrak{R}$) used for the modal paraconsistent operator $\neg$ equal to $\Box_P \neg_C$ where $\neg_C$ is classical negation.

Thus, here we will provide a slightly different Kripke semantics for the mZ logic, w.r.t. that in [1], Definition 6, but compatible with that used in [2]:

Definition 5: We define the Kripke model $\mathcal{M} = (\mathcal{W}, \sqsubseteq, \mathcal{R}, V)$ for the mZ logic, where $(\mathcal{W}, \sqsubseteq)$ is a poset, $\mathcal{R} \in \mathfrak{R}$ is an hereditary incompatibility binary accessibility relation for weakened paraconsistent negation with $R_{\Box_P} = ((\mathcal{W} \times \mathcal{W}) \backslash \mathcal{R})^{-1}$, and a mapping $V : (Var \bigcup \mathbf{2}) \times \mathcal{W} \to \mathbf{2}$, such that for any propositional letter $p \in Var$, if $w \sqsubseteq w'$ then $V(p,w) \leq V(p,w')$, such that $\forall w.(V(0,w) = 0$ and $V(1,w) = 1)$.

Then, for any world $w \in \mathcal{W}$ we define the satisfaction relation for any propositional formula A, denoted by $\mathcal{M} \models_w A$, as follows:

1. $\mathcal{M} \models_w p$ iff $V(p,w) = 1$, for any $p \in Var$.
2. $\mathcal{M} \models_w A \wedge B$ iff $\mathcal{M} \models_w A$ and $\mathcal{M} \models_w B$.
3. $\mathcal{M} \models_w A \vee B$ iff $\mathcal{M} \models_w A$ or $\mathcal{M} \models_w B$.
4. $\mathcal{M} \models_w A \Rightarrow B$ iff $\forall y((w \sqsubseteq y$ and $\mathcal{M} \models_y A)$ implies $\mathcal{M} \models_y B)$.
5. $\mathcal{M} \models_w \neg A$ iff $\forall y((w,y) \in R_{\Box_P}$ implies not $\mathcal{M} \models_y A)$.

$\Box$

Let us show that the intuitionistic negation $\neg_I$ may be obtained as a special case of the Birkhoff's polarity as well (as is our paraconsistent negation).

Corollary 2: For the incompatibility relation defined by $\mathcal{R} = (\mathcal{W} \times \mathcal{W}) \backslash R_\Box^{-1}$, where $R_\Box$ is a reflexive and transitive accessibility relation of the intuitionistic logic, from the Birkhoff's polarity method we obtain exactly the intuitionistic negation.

Proof: we have that $R_\Box = ((\mathcal{W} \times \mathcal{W})\backslash\mathcal{R})^{-1}$, and from Birkhoff's polarity,

$\|\neg A\| = \lambda(\|A\|) = \{x \in \mathcal{W} \mid \forall y \in \|A\|.(y, x) \in \mathcal{R}\}$, i.e,

$\mathcal{M} \vDash_x \neg A$ iff $\forall y(\mathcal{M} \vDash_y A$ implies $(y, x) \in \mathcal{R})$

iff $\forall y((x, y) \in R_\Box$ implies not $\mathcal{M} \vDash_y A)$ iff $\mathcal{M} \vDash_x \Box\neg_C A$

iff $\mathcal{M} \vDash_x \neg_I A$.

□

Proposition 1 and Corollary 2 demonstrate the general constructive approach to the paraconsistent negations based on the construction of an incompatibility relation with Birkhoff's polarity, and show that in this very general framework, the non-paraconsistent intuitionistic logic is only a special particular case.

IV. Philosophical and structural considerations of mZ logic

The recent history of the development of substructural relevant logics has an important example of reduction of Classical Propositional Calculus (CPC) in a number of its substructural logics, called intermediate propositional logics as well.

In this framework of progressively weakening of CPC we obtained a lattice of the intermediate logics, where the top element of this lattice was CPC and the bottom element the Intuitionistic Propositional Calculus (IPC).

The fundamental distinguishing characteristic of *intuitionism* is its interpretation of what it means for a mathematical statement to be *true*. In Brouwer's original intuitionism, the truth of a mathematical statement is a subjective claim: a mathematical statement corresponds to a mental construction, and a mathematician can assert the truth of a statement only by verifying the validity of that construction by intuition.

To an intuitionist, the claim that an object with certain properties exists is a claim that an object with those properties can be constructed. Any mathematical object is considered to be a product of a construction of a mind, and therefore, the existence of an object is equivalent to the possibility of its construction. This contrasts with the classical approach, which states that the existence of an entity can be proved by refuting its non-existence. For the intuitionist, this is not valid; the refutation of the non-existence does not mean that it is possible to find a construction for the putative object, as is required in order to assert its existence.

As such, intuitionism is a variety of mathematical constructivism; but it is not the only kind. Regardless of how it is interpreted, intuitionism does not equate the truth of a mathematical statement with its provability. However, because the intuitionistic notion of truth is more restrictive than that of classical mathematics, the intuitionist must reject some assumptions of classical logic to ensure that everything he/she proves is in fact intuitionistically true. This gives rise to intuitionistic logic.

This "weakening" from CPC into IPC can be seen as weakening of the interdependence of the basic for logic connectives: negation $\neg$ (unary operation) and binary operations, conjunction $\wedge$, disjunction $\vee$, and implication $\Rightarrow$. In order to distinguish these connectives ($\neg$ and $\Rightarrow$ only) in this hierarchy of different propositional logics, we will label them by a kind of logics: by a label C for the CPC and label I for the IPC, while the logic connectives of mZ will remain unlabeled.

In fact, if we consider these three logics (the intermediate logics between IPC and CPC are well understood and studied already), this weakening of logic connectives can be summarized as follows:

- CPC: here we have only to independent operators: the negation $\neg_C$ and one (usually taken implication) of the three binary operators. Another two operators are only derived operators.
- IPC: here we have the three mutually independent operators, $\wedge, \vee$ and $\Rightarrow$, while negation operator $\neg_I$ is derived one such that $\neg_I A$ is logically equivalent to the formula $A \Rightarrow 0$, where 0 is a contradiction (falsum) constant.
- mZ: here we have generally all four operators mutually independent, where the negation $\neg$ is obtained as a weakening of the intuitionistic negation $\neg_I$ by preserving its two fundamental properties: antitonicity and modal additivity.

This kind of weakening of the interdependence of the logic operators is a kind of obtaining of more powerful and relevant logics, by progressively extending classical logic with more powerful semantics: IPC can be seen as a means of extending classical logic with constructive semantics, so that IPC $\prec$ CPC, where $\preceq$ is the ordering in the current lattice of intermediate logics.

In what follows we will see that mZ logic can be seen as means of extending IPC with paraconsistent semantics as well. Consequently, from the philosophical point of view, mZ logic will become a new bottom element of the lattice of intermediate logics, that is mZ $\prec$ IPC.

Consequently, mZ extends a constructive logic IPC with da Costa paraconsistent semantics. In this hierarchy, we obtain more and more relevant logics (where the number of derivable theorems is progressively diminished), in the way that mZ is more relevant logic than IPC, and IPC is more relevant logic than all intermediate logics, while CPC is non relevant at all.

In this new point of view, the relationship between intuitionistic and paraconsistent logics is rather new one and in apparent opposition with the previous historical approach, where these two approaches are considered "dual", and are seen as two opposite extremes.

Theorem 1: All negative axioms in mZ logic are the theorems in IPC. Consequently, mZ $\prec$ IPC is a strict constructive paraconsistent weakening of the intuitionistic logic.

Proof: 1. From definition of the intutitionistic negation, $(A \Rightarrow 0) \equiv \neg_I A$, it holds that $(A \Rightarrow 0) \Rightarrow \neg_I A$ and by substituting A with 0 (denoted by $A \mapsto 0$) we obtain

1.1 $(0 \Rightarrow 0) \Rightarrow \neg_I 0$,

1.2 $A \Rightarrow A$, [theorem in IPC]

1.3 $0 \Rightarrow 0$, [1.2, and substitution $A \mapsto 0$]

1.4 $\neg_I 0$, [1.1, 1.3, (MP)]

1.5 $\neg_I 0 \Rightarrow (1 \Rightarrow \neg_I 0)$, [axiom (1) IPC$^+$, with $A \mapsto \neg_I 0$, $B \mapsto 1$]

(10.b) $1 \Rightarrow \neg_I 0$, [1.4, 1.5, (MP)]

1.6 $(1 \Rightarrow (A \Rightarrow 1))$, [axiom (1) IPC$^+$, with $A \mapsto 1$, $B \mapsto A$]

1.7 theorem 1, [theorem in IPC]

(11.b) $A \Rightarrow 1$, [1.6, 1.7, (MP)]

2. (11.b) $0 \Rightarrow A$ is an axiom in IPC.

3. Let us show that the weak contraposition (9.b), $(A \Rightarrow B) \Rightarrow (\neg_I B \Rightarrow \neg_I A)$, is a theorem in IPC:

3.1. The formula (T2), $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$, is a theorem in IPC.

3.2. By substitution of C with 0 in (T2), we obtain $(A \Rightarrow B) \Rightarrow ((B \Rightarrow 0) \Rightarrow (A \Rightarrow 0))$, that is, from the fact that $\neg_I B$ in IPC is defined by $B \Rightarrow 0$, we obtain that (9.b) is a theorem in IPC.

4. From the IPC+ theorem $((A \vee B) \Rightarrow C) \equiv ((A \Rightarrow C) \wedge (B \Rightarrow C))$, by substitution of C by 0 and by definition of intuitionistic negation $\neg_I$, we obtain $\neg_I (A \vee B) \equiv (\neg_I A \wedge \neg_I B)$ (modal additive property for intuitionistic negation), thus the theorem:

(12.b) $\neg_I (A \vee B) \Rightarrow (\neg_I A \wedge \neg_I B)$.

□

Notice that, as in mZ system, also in IPC, the intuitionistic negation $\neg_I$ is an additive modal operator with Birkhoff's polarity semantics given in Corollary 2.

Corollary 3: The relationship between the intuitionistic negation $\neg_I$ and the weakened paraconsistent negation $\neg$ in mZ logic is: $\neg_I \leq \neg$.

Proof: The meaning of $\neg_I \leq \neg$ in the Heyting algebra $((\mathcal{B}, \leq), \wedge, \vee, \Rightarrow)$, extended by the modal paraconsistent negation $\neg : \mathcal{B} \to \mathcal{B}$, is expressed by the sentence $(\forall x \in \mathcal{B})(\neg_I x \leq \neg x)$ or, equivalently in mZ logic, by the theorem $\neg_I A \Rightarrow \neg A$ for any proposition A.

By considering that $\neg_I A$ is a formula $A \Rightarrow 0$, it means that we have to show that a formula $(A \Rightarrow 0) \Rightarrow \neg A$ is a theorem as follows:

1 $(A \Rightarrow 1) \Rightarrow ((A \Rightarrow (1 \Rightarrow C)) \Rightarrow (A \Rightarrow C))$, [axiom (2) IPC$^+$, where $B \mapsto 1$]

2 $(A \Rightarrow (1 \Rightarrow C)) \Rightarrow (A \Rightarrow C)$, [1, (11,b),(MP)]

3 $((1 \Rightarrow C) \Rightarrow (1 \Rightarrow C)) \Rightarrow ((1 \Rightarrow C) \Rightarrow C)$, [2, where $A \mapsto 1 \Rightarrow C$]

4 $(1 \Rightarrow C) \Rightarrow C$, [3, $(1 \Rightarrow C) \Rightarrow (1 \Rightarrow C)$, (MP)]

5 $(1 \Rightarrow \neg A) \Rightarrow \neg A$, [4, where $C \mapsto \neg A$)]

6 $\neg A \equiv (1 \Rightarrow \neg A)$, [5, axiom (1) IPC$^+$ with $B \mapsto 1$]

7 $(A \Rightarrow 0) \Rightarrow (\neg 0 \Rightarrow \neg A)$, [from axiom (9.b), where $B \mapsto 0$]

8 $(A \Rightarrow 0) \Rightarrow (1 \Rightarrow \neg A)$, [$\neg 0 \equiv 1$, from (10.b) and (11.b) with $A \mapsto \neg 0$]

9 $(A \Rightarrow 0) \Rightarrow \neg A$, [from 8 and 5)]

that is, $\neg_I A \Rightarrow \neg A$.

□

Now we will show which axioms are necessary to add to mZ logic in order to obtain the intuitionistic logic:

Proposition 2: The intuitionistic logic IPC is equal to
mZ + $(\neg A \Rightarrow (A \Rightarrow 0))$.

Consequently, both formulae, $A \vee \neg A$, $\neg(A \wedge \neg A)$, excluded-middle and paraconsistent non-contradiction relatively, are not valid schemas in mZ logic.

Proof: The IPC is defined by eight positive axioms in IPC$^+$, plus:

(11.b) $0 \Rightarrow A$, falsity axiom,

plus two axioms for the intuitionistic negation,

(N1) $\neg A \Rightarrow (A \Rightarrow 0)$,

(N2) $(A \Rightarrow 0) \Rightarrow \neg A$.

Consequently, it is enough only to show that $(A \Rightarrow 0) \Rightarrow \neg A$ is a theorem in mZ.

From the weak contraposition axiom schemata (9.b) in the case when B is substituted by 0, we obtain that $(A \Rightarrow 0) \Rightarrow (\neg 0 \Rightarrow \neg A)$ and from the fact that $\neg 0 \equiv 1$ (from (11.b) when A is substituted by $\neg 0$ and (10.b)), we obtain $(A \Rightarrow 0) \Rightarrow (1 \Rightarrow \neg A)$.

It is enough to show that $(1 \Rightarrow \neg A) \equiv \neg A$ is a theorem in mZ, and it is demonstrated in point 6 of the proof of Corollary 3.

Thus, we have that the non-contradiction schema is not a valid schema in mZ (Lemma 3 in [2]). Moreover, if we denote the set of theorems of IPC by $\mathcal{L}_{ICP}$ and the set of theorems of mZ by $\mathcal{L}_{mZ_n}$, from the fact that $\mathcal{L}_{mZ_n} \subset \mathcal{L}_{ICP}$ and from the fact that the excluded-middle schema does not hold in IPC, i.e., $A \vee \neg A \notin \mathcal{L}_{ICP}$, we also obtain $A \vee \neg A \notin \mathcal{L}_{mZ_n}$.

□

Thus, based on Theorem 1 and Proposition 2, we obtain that mZ is constructive logic as IPC (they have the same set of theorems for their positive fragment), but the set of theorems with negation operator $\neg$ in mZ is a strict paraconsistent *subset* of theorems with intuitionistic negation $\neg_I$ in IPC.

Consequently, mZ is a *subintuitionistic* logic. So, mZ is a more useful logic than IPC, that is more relevant w.r.t. the IPC, because it avoids explosive inconsistency.

From the point of da Costa paraconsistency of the mZ logic, it was demonstrated in [2] (Theorem 11) that the da Costa axioms (11) and (12) are theorems in mZ, so that in mZ all hierarchy of da Costa's systems are present in this single mZ logic: thus mZ is by itself da Costa paraconsistent and its axioms implicitly cover the da Costa's reductio relativization property (and its propagation) and combines it with the constructive property of the intuitionistic logics.

These results demonstrate that the paraconsistency is not simply dual to the constructivism, as was historically supposed and investigated. In fact, mZ combines both constructive and paraconsistent properties, where both excluded-middle and the non-contradiction schemas are not valid.

We have demonstrated that the paraconsistency is based on a very constructive approach, and we will show it in the rest of this paper by the construction of incompatibility relations for the paraconsistent negations: we will present a number of useful subintuitionistic mZ logics for practical applications as well.

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V. Paraconsistent Logic Programming

In what follows we will present a number of examples of many-valued logics that are members of the mZ system. In particulary, we will exam the logics with a minimal cardinality (three and four-valued logics) and their applications in Paraconsistent Logic Programming. Because of that, we will shortly introduce some concepts of the 4-valued bilattices and fixpoint semantics for logic programs.

So far, research in many-valued logic programming has proceeded along different directions: *Signed* logics [22], [23] and *Annotated* logic programming [24], [25], [26] which can be embedded into the first, *Bilattice-based* logics [27], [28], and *Quantitative rule-sets* [29]. Earlier studies of these approaches quickly identified various distinctions between these frameworks. For example, one of the key insights behind bilattices was the interplay between the truth values assigned to sentences and the (non classic) notion of *implication* in the language under considerations. Thus, rules (implications) had weights (or truth values) associated with them as a whole. The problem was to study how truth values should be propagated "across" implications.

Annotated logics, on the other hand, appeared to associate truth values with each component of an implication rather than the implication as a whole. Roughly, based on the way in which uncertainty is associated with facts and rules of a program, these frameworks can be classified into *implication based* (IB) and *annotation based* (AB).

In the IB approach a rule is of the form $A \leftarrow^{\alpha} B_1, .., B_n$, which says that the certainty associated with the implication is α. Computationally, given an assignment I of logical values to the B_is, the logical value of A is computed by taking the "conjunction" of logical values $I(B_i)$ and then somehow "propagating" it to the rule head A.

In the AB approach a rule is of the form $A : f(\beta_1, .., \beta_n) \leftarrow B_1 : \beta_1, ..., B_n : \beta_n$, which asserts "the certainty of the atom A is least (or is in) $f(\beta_1, .., \beta_n)$, whenever the certainty of the atom B_i is at least (or is in) β_i, $1 \leq i \leq n$", where f is an n-ary computable function and β_i is either constant or a variable ranging over many-valued logic values. The comparison in [30] shows:

1- while the way implication is treated on the AB approach is closer to the classical logic, the way rules are fired in the IB approach has definite intuitive appeal.

2- the AB approach is strictly more expressive than IB. The down side is that query processing in the AB approach is more complicated, e.g. the fixpoint operator is not continuous in general, while it is in the IB approaches.

From the above points discussed in [30], it is believed that IB approach is easier to use and is more amenable for efficient implementations.

The other problem is that the Fitting fixpoint semantics for IB logic programs, based exclusively on a bilattice-algebra operators, suffer two drawbacks:

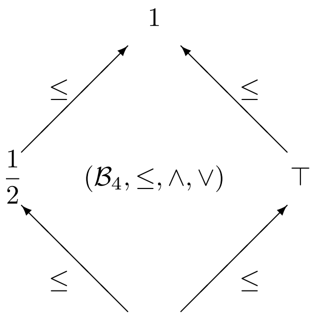
- The lack of the notion of tautology (bilattice negation operator is an *epistemic* negation) leads to difficulties in defining proof procedures and to the need for additional complex truth-related notions as "formula closure" [30];
- There is an unpleasant asymmetry in the semantics of implication (which is strictly 2-valued) w.r.t. all other bilattice operators (which produce any truth value from the bilattice) - it is a sign that strict bilattice language is not enough expressive for logic programming and we need some reacher (different) syntax for logical programming.

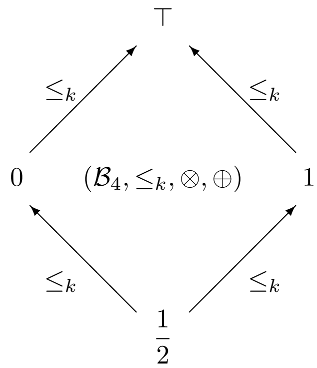
Both of these two drawbacks now can be avoided by using many-valued paraconsistent mZ logics, with a complete distributive lattice $\mathcal{B}_n$ of $n \geq 3$ logic truth values, such that $\{0,1\} \subseteq \mathcal{B}_n$ are the bottom and the top values respectively:

- The mZ logics have a well defined axiomatization and the notion of tautology, and the proof structures (the subset of designated elements in their matrix is a singleton $\{1\}$).
- The logic implication is intuitionistic one (relative-pseudocomplement) that is many-valued as conjunction and disjunction operators and satisfy a good constructive property, that is, for $\alpha, \beta \in \mathcal{B}_n$ we have that $(\alpha \Leftarrow \beta) = 1$ if $\alpha \geq \beta$.
  Consequently, if β is the logic value of a body of one logic program clause, with α derived logic value of the head of this clause, then the satisfaction of this clause is satisfied if $\alpha \geq \beta$. Thus, it express the principle of *propagation of truth* in logic programming clauses [31], [32], [33], [34] and the symbol $\leftarrow$ from the body into the head of a logic rule corresponds to the intuitionistic implication of the mZ system.

In [35], Belnap introduced a logic intended to deal in a useful way with inconsistent or incomplete information. It is the simplest example of a non-trivial bilattice and it illustrates many of the basic ideas concerning them. We denote the four logic values as $\mathcal{B}_4 = \{0, \frac{1}{2}, 1, \top\}$, where 1 is *true*, 0 is *false*, $\top$ is inconsistent (both true and false) or *possible*, and $\frac{1}{2}$ is *unknown*.

As Belnap observed, these values can be given two natural orders: *truth* order, $\leq$, and *knowledge* order, $\leq_k$, such that $0 \leq \top \leq 1$, $0 \leq \frac{1}{2} \leq 1$, and $\frac{1}{2} \leq_k 0 \leq_k \top$, $\frac{1}{2} \leq_k 1 \leq_k \top$.





Meet and join operators under $\leq$ are denoted $\wedge$ and $\vee$; they are natural generalizations of the usual conjunction and disjunction notions. Meet and join under $\leq_k$ are denoted $\otimes$ (*consensus*, because it produces the most information that two truth values can agree on) and $\oplus$ (*gullibility*, it accepts anything it's told). We have that:

$0 \otimes 1 = \frac{1}{2}$, $0 \oplus 1 = \top$, $\top \wedge \frac{1}{2} = 0$ and $\top \vee \frac{1}{2} = 1$.

There is an epistemic notion of truth negation, denoted by $\sim$, (reverses the truth $\leq$ ordering, while preserving the knowledge $\leq_k$ ordering): switching 0 and 1, leaving $\frac{1}{2}$ and $\top$. A more general information about bilattices may be found in [28]. The Belnap's 4-valued bilattice is infinitary distributive.

A (ordinary) Herbrand interpretation is a many-valued mapping $I : H_P \to \mathcal{B}_4$. If P is a many-valued logic program with the Herbrand base H_P then the ordering relations and operations in a bilattice $\mathcal{B}_4$ are propagated to the function space $\mathcal{B}_4^{H_P}$, that is the set of all Herbrand interpretations (functions) $I : H_P \to \mathcal{B}_4$.

Definition 6: Ordering relations are defined on the function space $\mathcal{B}_4^{H_P}$ pointwise, as follows: for any two Herbrand interpretations $I, I' \in \mathcal{B}_4^{H_P}$,

1. $I \leq I'$ if $I(A) \leq I'(A)$ for all $A \in H_P$.
2. $I \leq_k I'$ if $I(A) \leq_k I'(A)$ for all $A \in H_P$.
3. $\neg I$, such that $(\neg I)(A) = \neg(I(A))$.

□

This interpretation can be inductively extended into the map I^* to all ground formulae in the standard truth-functional way, for any two ground formulae (that is, without free variables) ϕ and ψ:

- $I^*(\phi) = I(\phi)$ for any ground atom (considered as a *proposition* in mZn as well) $\phi \in H_P$;
- $I^*(\phi \odot \psi) = I^*(\phi) \odot I^*(\psi)$ for $\odot \in \{\wedge, \vee, \Rightarrow\}$;
- $I^*(\neg\phi) = \neg(I^*(\phi))$;

It is straightforward [28] that this makes a function space $\mathcal{B}_4^{H_P}$ itself a complete infinitary distributive bilattice.

Definition 7: [28] Let P be a logic program, with P^* the set of all ground instances of members of P and a valuation $I : H_P \to \mathcal{B}_4$. We define the monotonic in $\leq_k$ immediate consequence operator $\Phi_P : \mathcal{B}_4^{H_P} \to \mathcal{B}_4^{H_P}$ such that for each $A \in H_P$,

1. if A is not the head of any member of P^*,
$\Phi_P(I)(A) = f$,
2. otherwise $\Phi_P(I)(A) = \bigvee\{I^*(B) \mid A \leftarrow B \ is \ in \ P^*\}$.

□

Fitting [28] has demonstrated that Φ_P is monotonic w.r.t. knowledge $\leq_k$ ordering and that the Knaster-Tarski theorem gives the smallest fixed point, which corresponds to the 4-valued stable models of the logic programs.

It is interesting to consider the minimal cardinality for such one many-valued logic of the system mZ_n:

Lemma 2: In the mZ calculus, the minimal cardinality is represented by the 3-valued logic $\mathcal{B}_3 = \{0, \frac{1}{2}, 1\}$ (a Belnap's sublattice obtained by elimination of the value $\top$) but with negation different from the well-known 3-valued Kleene logic.

We obtain a weakening of the Kleene-negation by the fact that, instead of $\neg 1 = 0$, we have that $\neg 1 = \frac{1}{2}$. It can be used for logic programming with Fitting's knowledge fixpoint-semantics in Definition 7.

Proof: Let us consider $\mathcal{W} = \{2, 3\}$ with $\sqsubseteq$ equal to inverse of the ordering of numbers and $\mathcal{H}_W = \{\emptyset, \{2\}, \{2, 3\}\}$ with the unique atom $\{2\}$ and with the isomorphism $is : (\mathcal{H}_W, \subseteq, \bigcap, \bigcup, \rightharpoonup, \lambda) \to (\mathcal{B}_3, \leq, \wedge, \vee, \Rightarrow, \neg)$ in Definition 4 such that $is(\emptyset) = 0$, $is(\mathcal{W}) = 1$ and $is(\{2\}) = \frac{1}{2}$ (unknown value), and hereditary relation $\mathcal{R} = \{(2, 2), (3, 2)\} \in \mathfrak{R}$.

It is easy to verify that $\lambda(\mathcal{W}) = \{2\} \neq \emptyset$ (i.e., from the algebra isomorphisms is, $\neg 1 = \frac{1}{2} \neq 0$) and that for $V = \{2\}$, $\lambda(V \bigcap \lambda(V)) = \{2\} \neq \mathcal{W}$ (i.e.,$\neg(\frac{1}{2} \wedge \neg\frac{1}{2}) = \frac{1}{2} \neq 1$), so that the principle of non-contradiction is valid.

It is different from the Kleene logic where $\neg 1 = 0$, while in our 3-valued paraconsistent mZ logic we have by the weakening of negation that $\neg 1 = \frac{1}{2}$ (i.e., $\neg 1 = \neg\frac{1}{2} = \frac{1}{2}, \neg 0 = 1$), but $\neg$ is monotonic w.r.t. the knowledge ordering $\leq_k$ so that this logic can be used for logic programming with Fitting's fixpoint-semantics and stable models.

□

Note that this paraconistent 3-valued logic satisfy all axioms of the mCZ logics as well.

Let us show that in mZ system there is a Belnap's bilattice based logic with weakened negation that is monotonic w.r.t the knowledge ordering $\leq_k$. So, we can use this 4-valued paraconsistent mZ logic for a logic programming, as explained above for the standard bilattice negation $\sim$.

Lemma 3: In the mZ calculus, the bilattice based logic is represented by the 4-valued Belnap's bilattice $\mathcal{B}_4 = \{0, \frac{1}{2}, 1, \top\}$, but with negation $\neg$ different from the original bilattice negation $\sim$. We obtain a weakening of the negation $\sim$ by the fact that, instead of $\sim 1 = 0$ and $\sim \top = \top$, we define $\neg 1 = \frac{1}{2}$ and $\neg\top = 1$ respectively.

This paracosistent bilattice-based mZ logic with weakened negation $\neg$ can be used for a logic programming with Fiting's fixpoint-semantics and stable models, in the same way as the Belnap's logic with the standard negation $\sim$.

Proof: Let us consider the poset $\mathcal{W} = \{a, b\}$, with $a \sqsubseteq a$ and $b \sqsubseteq b$, and the set of hereditary subsets $\mathcal{H}_W = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$, with the isomorphism $is : (\mathcal{H}_W, \subseteq, \bigcap, \bigcup, \rightharpoonup, \lambda) \to (\mathcal{B}_4, \leq, \wedge, \vee, \Rightarrow, \neg)$ where $0 = is(\emptyset)$ corresponds to the false value, $1 = is(\mathcal{W})$ to the true value, $is(\{a\}) = \top$, $is(\{b\}) = \frac{1}{2}$ to the unknown value, and hereditary relation $\mathcal{R} = \{(a, a), (b, b), (a, b)\} \in \mathfrak{R}$.

It is easy to verify that $\lambda(\mathcal{W}) = \{b\} \neq \emptyset$ and for $V = \{b\}$, $\lambda(V \bigcap \lambda(V)) = \{b\} \neq \mathcal{W}$, so that the principle of non-contradiction is valid.

We have that $\lambda(\{b\}) = \{b\}$ and $\lambda(\{a\}) = \{a, b\}$.
So, we obtain that $\neg 1 = \neg\frac{1}{2} = \frac{1}{2}$ and $\neg 0 = 1$ and $\neg\top = 1$.

Thus, it is easy to verify that the obtained weakened negation $\neg$ is monotonic w.r.t. the knowledge ordering $\leq_k$ and we are able to use the fix-point semantics in Definition 7 for the logic programs.

□

Note that if we omit the 4-th value $\top$ from the paraconsistent Belnap's logic in Lemma 3, that is, replace it by $\frac{1}{2}$, we obtain, as reduction, the paraconsistent Kleene's logic in Lemma 2, as my be seen from their truth-value tables:

	IPC	mZ
0	1	1
$\frac{1}{2}$	$\top$	$\frac{1}{2}$
1	0	$\frac{1}{2}$
$\top$	$\frac{1}{2}$	1

$\mapsto$

	IPC	mZ
0	1	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$

VI. Family of (in)finitary canonical constructions

Notice that the example in Lemma 2 may be generalized to all sublattices $(\mathcal{H}_W, \subseteq, \bigcap, \bigcup)$ (of the powerset lattice $(\mathcal{P}(\mathcal{W}), \subseteq, \bigcap, \bigcup)$) of all hereditary subsets of the total ordering $(\mathcal{W}, \sqsubseteq)$, by defining an antitonic negation operator such that:
$\lambda(\emptyset) = \mathcal{W}$ and $\lambda(\mathcal{W})$ is equal to a singleton set $*$ (it is an *atom* in $\mathcal{H}_W$).

That is, $\neg 0 = 1$ and $\neg 1 = a \in \mathcal{B}$ has a fixed value $a = is(*) > 0$, where, from Definition 4,

$is : (\mathcal{H}_W, \subseteq, \bigcap, \bigcup, \rightharpoonup, \lambda) \rightarrow (\mathcal{B}, \leq, \wedge, \vee, \Rightarrow, \neg)$
is the isomorphism such that $is(\emptyset) = 0$ and $is(\mathcal{W}) = 1$. Thus, we need a many-valued framework.

For a poset $(\mathcal{W}, \sqsubseteq)$ and an element $x \in \mathcal{W}$, we denote the downward closed subset $\{y \in \mathcal{W} \mid y \sqsubseteq x\}$ by $\downarrow x$, and the upward closed subset $\{y \in \mathcal{W} \mid x \sqsubseteq y\}$ by $\uparrow x$.

It is interesting to show how we are able to make a *canonical* construction of any n-valued logic ($n \geq 4$) that is a member of the mZ system, useful for practical applications.

Example: Let us consider, for example, the following construction (infinite as well):

Lemma 4: Let us define $\mathcal{W} = \{2, 3, 4, ..., n\}$, where $n \geq 4$ can be finite or infinite ($n = \infty$), with $\sqsubseteq$ equal to the inverse of the ordering of numbers and hence $\mathcal{W} = \uparrow n$ and $\uparrow 2 = \{2\}$. Then we define the set of hereditary subsets:

$\mathcal{H}_W = \{\emptyset, \uparrow 2, \uparrow 3, \uparrow 4, ..., \mathcal{W}\}$ and the hereditary relation $\mathcal{R} = \bigcup_{j \in \mathcal{W}} (B_{j,1} \times B_{j,2}) \in \mathfrak{R}$, where $B_{j,1} = \uparrow j \in \mathcal{H}_W$, $B_{j,2} = \uparrow (n - j + 2) \in \mathcal{H}_W$.

The algebra of hereditary subsets $(\mathcal{H}_W, \subseteq, \bigcap, \bigcup, \rightharpoonup, \lambda)$ is an algebraic representation of the paraconsistent n-valued mZ logic.

Proof: We have that $\lambda(\emptyset) = \mathcal{W}$, while for each $V = \uparrow j \in \mathcal{H}_W$ (i.e., $V \neq \emptyset$), $\lambda(V) = \uparrow (n - j + 2) \in \mathcal{H}_W$. We can define the n-valued lattice

$(\mathcal{B}_n, \leq, \wedge, \vee, 0, 1) = \{0, \frac{1}{n-1}, ..., \frac{1}{2}, 1\}$,
where $\wedge$ (meet) and $\vee$ (join) are the functions *min* and *max* respectively, for this n-valued mZ logic with the isomorphisms $is : \mathcal{H}_W \rightarrow \mathcal{B}_n$ such that $is(\emptyset) = 0$ and $is(\uparrow k) = \frac{1}{n-k+1}$ for $k = 2, ..., n$, and hence $is(\mathcal{W}) = is(\uparrow n) = 1$.

So, we have that $\lambda(\mathcal{W}) = \lambda(\uparrow n) = \uparrow 2 = \{2\} \neq \emptyset$ (we recall that $\uparrow 2 = \{2\}$ is the atom in the complete distributive lattice $(\mathcal{H}_W, \subseteq)$), so that $\neg 1 = is \circ \lambda \circ is^{-1}(1) = is(\lambda(\mathcal{W})) = is(\uparrow 2) = \frac{1}{n-1} \neq 0$ which satisfies the fact that we eliminated the axiom $\neg 1 \Rightarrow 0$ (and theorem $\neg 1 = 0$) from our system mZ.

Hence, for paraconsistent negation we have $\neg \frac{1}{k} = \frac{1}{n-k}$ and $\neg\neg$ identity, for $k = 1, 2, ..., n-1$ (however, with $\neg\neg 0 = \frac{1}{n-1} \neq 0$).
Moreover, for each $V = \uparrow j \in \mathcal{H}_W$, for $3 \leq j \leq n-1$,

$\lambda(V \bigcap \lambda(V)) = \lambda(\uparrow j \bigcap \uparrow (n - j + 2))$
$= \lambda(\uparrow min(j, n - j + 2)) \neq \mathcal{W}$,
(i.e., $\neg(\frac{1}{n-j+1} \wedge \neg \frac{1}{n-j+1}) = \neg(min\{\frac{1}{n-j+1}, \frac{1}{j-1}\})$
$= max\{\frac{1}{n-j+1}, \frac{1}{j-1}\} \neq 1$
so that for each proposition A whose 'logic value' is a $V \neq \emptyset$ and $V \neq \mathcal{W}$ (set of possible worlds in $\mathcal{W}$ in the Kripke-like semantics where A is true) we have that $\neg(A \wedge \neg A)$ is not true, and the schema $\neg(A \wedge \neg A)$ is not valid in this n-valued logic.

Notice that the 3-valued logic of the Lemma 2 *is not* a canonical construction. In fact, in the canonical solution above we would obtain the *different* incompatibility hereditary relation $\mathcal{R} = \{(2, 2), (3, 2), (2, 3)\}$, but in this case $\neg(A \wedge \neg A)$ would be valid, thus NdC1 not satisfied.

End example
$\square$

However, in what follows we will use another more *general canonical* construction.

The approach used in the example above is a particular application (for complete distributive lattices of *total* orderings) of the *autoreferential* one [10], [33], based on the observation that the set of 'logic values' in $\mathcal{H}_W$ is a complete distributive lattice.

Thus, the construction of the incompatibility relation is fundamentally autoreferential (w.r.t. the set of logic values of a many-valued logic). Consequently, this approach can be used to model *any* desired split negation (antitonic and additive-modal unary operator), for example as follows.

In what follows we will use the method of autoreferential semantics for representation of many-valued algebras [10] corresponding to a number of different logics as examples in this section. So, from [10] for *any* (also non totally ordered as that in Lemma 4 of a canonical construction) complete distributive bounded latice of truth-values $(\mathcal{B}, \leq, \wedge, \vee, 0, 1)$, there is the 0-Lifted Birkhoff isomorphism with its set-based lattice representation:

$\downarrow^+ : (\mathcal{B}, \leq, \wedge, \vee) \simeq (\mathcal{B}^+, \subseteq, \bigcap, \bigcup)$
where by $\mathcal{W} \subset \mathcal{B}$ we denote the subset of join-irreducible elements (with bottom element $0 \notin \mathcal{W}$ because it is not join-ireducible) in the complete distributive lattice of truth values $(\mathcal{B}, \leq, \wedge, \vee)$, and we define $\widehat{\mathcal{B}} =_{def} \mathcal{W} \bigcup \{0\}$, so that we define the hereditary subset by

$\downarrow^+ x \;\; = \;\; \{y \in \mathcal{B} \mid y \leq x\} \bigcap \widehat{\mathcal{B}}$
with the set of all downward closed hereditary subsets $\mathcal{B}^+ = \{\downarrow^+ a \mid a \in \mathcal{B}\} \subseteq \mathcal{P}(\mathcal{B})$, so that

$\downarrow^+ \bigvee = id_{\mathcal{B}^+} : \mathcal{B}^+ \rightarrow \mathcal{B}^+$ and $\bigvee \downarrow^+ = id_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{B}$.
Thus, the operator $\downarrow^+$ is the inverse of the supremum operation $\bigvee : \mathcal{B}^+ \rightarrow \mathcal{B}$.

The Heyting's and multi-modal extensions (nor example, for modal paraconsistent negation operators in our case) of these complete distributive latices are provided in [10], with their representation by the isomorphism between many-valued Heyting algebra of truth-values in $\mathcal{B}$ (with implication defined as pseudocomplement

$x \rightharpoonup y = \bigvee\{z \in \mathcal{B} \mid z \wedge x \leq y\}$
and negation by $\neg x = x \rightharpoonup 0$) and its set-based representation algebra:

$$\downarrow^+ : (\mathcal{B}, \leq, \wedge, \vee, \neg, \Rightarrow, 0, 1) \;\simeq\; (\mathcal{B}^+, \subseteq, \cap, \cup, \widehat{\neg}, \rightharpoonup, \{0\}, \widehat{\mathcal{B}}) \quad (2)$$

where the set-based implication operator is defined by

$(\downarrow^+ x) \rightharpoonup (\downarrow^+ y) \;=\; \downarrow^+ (x \Rightarrow y)$
Thus, what we need is only to replace the intuitionistic negation used in Heyting algebras by their paraconsistent weakening (substitute $\widehat{\neg}$ by λ in the isomorphism above),

and hence to find the incompatibility relation $\mathcal{R}$ used in Birkhoff-polarity for the split negation λ, in order to preserve this representation-isomorphism above, and to obtain that the set-based algebra in (2) be just the algebra on hereditary subsets $(\mathcal{H}_W, \subseteq, \bigcap, \bigcup, \rightarrowtail, \lambda)$ as provided by Definition 4.

However, the bottom element of $\mathcal{H}_W$ is the empty set $\emptyset$ while the bottom element of $\mathcal{B}^+$ is the singleton set $\{0\}$, and hence here we need a different representation from that in (2). So, we define this autoreferential semantics for the (inverse w.r.t isomorphism in (2)) isomorphism in Definition 4:

Definition 8: The autoreferential semantics for the required isomorphism (1) in Definition 4 is specified as follows:

1. The set of possible worlds $\mathcal{W}$ is just the set of join-irreducible elements in the lattice of truth-values $\mathcal{B}$;
2. The set of hereditary subsets of join-irreducible elements
$$\mathcal{H}_W =_{def} \{\downarrow_0^+ x \mid x \in \widehat{\mathcal{B}} = \mathcal{W} \cup \{0\}\} \qquad (3)$$
where $\downarrow_0^+ x =_{def} (\downarrow^+ x) \backslash \{0\}$, and $\backslash$ is the set-substraction operation.
3. For each hereditary subset $S \in \mathcal{H}_W$, we define the isomorphism (1) in Definition 4 by
$$is(S) =_{def} \widehat{\bigvee} S \qquad (4)$$
where $\widehat{\bigvee} S =_{def} \bigvee\{S \cup \{0\}\}$, in order to obtain the inverse isomorphism $is^{-1} = \downarrow_0^+$.

That is, we obtain the isomorphism

$$\downarrow_0^+ : (\mathcal{B}, \leq, \wedge, \vee, \Rightarrow, \neg, 0, 1) \simeq (\mathcal{H}_W, \subseteq, \cap, \cup, \rightharpoonup, \lambda, \emptyset, \mathcal{W}) \qquad (5)$$

and hence, by this homomorphisam, $\downarrow_0^+ \neg = \lambda \downarrow_0^+$, i.e.,

$\lambda = \downarrow_0^+ \neg \widehat{\bigvee} : \mathcal{H}_W \to \mathcal{H}_W$.

□

Notice that the example in Lemma 2 may be generalized to all sublattices $(\mathcal{H}_W, \subseteq, \bigcap, \bigcup)$ (of the powerset lattice $(\mathcal{P}(\mathcal{W}), \subseteq, \bigcap, \bigcup)$) of all hereditary subsets of the ordering $(\mathcal{W}, \sqsubseteq)$, by defining an antitonic negation operator such that:

$\lambda(\emptyset) = \mathcal{W}$ and

$\lambda(\mathcal{W})$ is a singleton set $*$ (it is an *atom* in $\mathcal{H}_W$)

That is, $\neg 0 = 1$ and $\neg 1 = a \in X$ has a fixed value $a = is(*) > 0$, where, from Definition 4,

$is : (\mathcal{H}_W, \subseteq, \bigcap, \bigcup, \rightharpoonup, \lambda) \to (\mathcal{B}, \leq, \wedge, \vee, \Rightarrow, \neg)$

is the isomorphism such that $is(\emptyset) = 0$ and $is(\mathcal{W}) = 1$. Thus, we need a many-valued framework.

The approach used in this canonical construction is a particular application (for complete distributive bounded lattices $(\mathcal{B}, \leq, \wedge, \vee, 0, 1)$) of the *autoreferential* one [10], [33], based on the observation that $\mathcal{H}_W$ is a complete distributive lattice. Thus, the construction of the incompatibility relation is fundamentally autoreferential (w.r.t. the set of logic values of a many-valued logic). Consequently, this approach can be used to model *any* desired split negation (antitonic and additive-modal unary operator):

Proposition 3: For a given negation (antitonic additive modal operator) $\neg$, we are able to define the hereditary-incompatibility relation by

$$\mathcal{R} = \bigcup_{a \in \mathcal{W}} (\downarrow_0^+ a) \times (\downarrow_0^+ \neg a) \qquad (6)$$

in order to define the split negation λ which satisfies the autoreferential semantics isomorphism (5),

$\downarrow_0^+ : (\mathcal{B}, \leq, \wedge, \vee, \Rightarrow, \neg, 0, 1) \simeq (\mathcal{H}_W, \subseteq, \cap, \cup, \rightharpoonup, \lambda, \emptyset, \mathcal{W})$

that is, the condition

$$\lambda = \downarrow_0^+ \neg \widehat{\bigvee} : \mathcal{H}_W \to \mathcal{H}_W \qquad (7)$$

Proof: For the case when $S = \emptyset \in$ we obtain the banal result that $\lambda\emptyset = \mathcal{W} \in \mathcal{H}_W$, so that we are interested only for non-empty hereditary subsets of join-irreducible elements from (3), $S = \downarrow_0^+ x \in \mathcal{H}_W$ for some $x \in \mathcal{B}$, and hence we have that

$\lambda S =_{def} \{y \in \mathcal{B} \mid \forall u \in S.(u, y) \in \mathcal{R}\}$

$= \{y \in \mathcal{B} \mid \forall u \in (\downarrow_0^+ x).(u, y) \in \mathcal{R}\}$

$\supseteq \quad (\downarrow_0^+ \neg x)$ from (6) and $x \in \mathcal{W}$.

Let us show that we obtain strictly the result $\lambda S = (\downarrow_0^+ \neg x)$, that is, that we have no any contribution to λS of another components $(\downarrow_0^+ a) \times (\downarrow_0^+ \neg a) \subset \mathcal{R}$ in (6). In order to give such a contribution to λS, from definition of λ we must have that $S \subset (\downarrow_0^+ a)$ and from the fact that $S = (\downarrow_0^+ a)$ it is possible only if $x < a$, that is $\neg x \geq \neg a$ and hence $(\downarrow_0^+ \neg x) \supseteq (\downarrow_0^+ \neg a)$. Consequently, any $y \in (\downarrow_0^+ \neg a)$ is already in $(\downarrow_0^+ \neg x)$ as well, and we can not have any new contribution to λS, that is, we obtain the strict result that for each $x \in \mathcal{B}$

$(\downarrow_0^+ \neg x) = \lambda S = \lambda(\downarrow_0^+ x)$

and hence we obtain that $\downarrow_0^+ \neg = \lambda \downarrow_0^+$, so that $\lambda = \lambda(\downarrow_0^+ \widehat{\bigvee}) = (\downarrow^+ \neg)\widehat{\bigvee}$.

This completes the proof.

□

This proposition is very useful because it offers a method for transformation of any many-valued logic based on Heyting algebras (thus with intuitionistic implication) into da Costa paraconsistent mZ logic by simply weakening of the negation operation (as we will see in what follows in a number of examples) and to offer a Kripke semantics for them (in that case we see this paraconsistent logic as a bimodal two-valued logic).

Let us consider the previous examples of mZ logic for 3-valued and 4-valued logics by using the autoreferential semantics:

- Da Costa paraconsistent Kleene logic in Lemma 2. The set of join-irreducible elements is $\mathcal{W} = \{\frac{1}{2}, 1\}$ with $\sqsubseteq$ equal to inverse truth-ordering $\leq^{-1}$ and $\widehat{\mathcal{B}} = \mathcal{W} \bigcup \{0\} = \{0, \frac{1}{2}, 1\} = \mathcal{B}_3$, and hence from (3), $\mathcal{H}_W = \{\emptyset, \{\frac{1}{2}\}, \mathcal{W}\}$ with the isomorphism
$is : (\mathcal{H}_W, \subseteq, \bigcap, \bigcup, \rightharpoonup, \lambda) \to (\mathcal{B}_3, \leq, \wedge, \vee, \Rightarrow, \neg)$
in Definition 4, such that $is(\emptyset) = 0$, $is(\mathcal{W}) = 1$ and $is(\{\frac{1}{2}\}) = \frac{1}{2}$ (unknown value), and hereditary relation, from (6), is given by $\mathcal{R} = \{(\frac{1}{2}, \frac{1}{2}), (1, \frac{1}{2})\} \in \mathfrak{R}$ for the split negation λ.
It is easy to verify that $\lambda(\mathcal{W}) = \{\frac{1}{2}\} \neq \emptyset$ (i.e., from

the algebra isomorphisms is, $\neg 1 = \frac{1}{2} \neq 0$) and that for $V = \{\frac{1}{2}\}$,

$$\lambda(V \bigcap \lambda(V)) = \{\frac{1}{2}\} \neq \mathcal{W}$$

(i.e.,$\neg(\frac{1}{2} \wedge \neg\frac{1}{2}) = \frac{1}{2} \neq 1$), so that the principle of non-contradiction is valid.

It is different from the Kleene logic where $\neg 1 = 0$, while in our 3-valued paraconsistent mZ logic we have by the weakening of negation that $\neg 1 = \frac{1}{2}$ (i.e., $\neg 1 = \neg\frac{1}{2} = \frac{1}{2}, \neg 0 = 1$).

- Da Costa paraconsistent Belnap bilattice logic in Lemma 3. The set of join-irreducible elements is $\mathcal{W} = \{\frac{1}{2}, \top\}$ with $\sqsubseteq$ equal to inverse truth-ordering $\leq^{-1}$ and $\widehat{\mathcal{B}} = \mathcal{W} \bigcup \{0\}$, and hence from (3) $\mathcal{H}_W = \{\emptyset, \{\top\}, \{\frac{1}{2}\}, \mathcal{W}\}$, with the isomorphism

  $$is : (\mathcal{H}_W, \subseteq, \bigcap, \bigcup, \rightharpoonup, \lambda) \rightarrow (\mathcal{B}_4, \leq, \wedge, \vee, \Rightarrow, \neg)$$

  where $0 = is(\emptyset)$ corresponds to the false value, $1 = is(\mathcal{W})$ to the true value, $is(\{\top\}) = \top$, $is(\{\frac{1}{2}\}) = \frac{1}{2}$ to the unknown value, and hereditary relation,from (6),

  $$\mathcal{R} = \{(\top, \top), (\tfrac{1}{2}, \tfrac{1}{2}), (\top, \tfrac{1}{2})\} \in \mathfrak{R}.$$

  It is easy to verify that $\lambda(\mathcal{W}) = \{\frac{1}{2}\} \neq \emptyset$ and for $V = \{\frac{1}{2}\}$, $\lambda(V \bigcap \lambda(V)) = \{\frac{1}{2}\} \neq \mathcal{W}$, so that the principle of non-contradiction is valid. We have that

  $$\lambda(\{\tfrac{1}{2}\}) = \{\tfrac{1}{2}\} \text{ and } \lambda(\{\top\}) = \mathcal{W} = \{\tfrac{1}{2}, \top\}.$$

  Consequently, we obtain that $\neg 1 = \neg\frac{1}{2} = \frac{1}{2}$ and $\neg 0 = 1$ and $\neg\top = 1$.

Based on these two examples and on the autoreferential semantics [10], [33], used for a construction of $(\mathcal{W}, \sqsubseteq)$ and incompatibility relation in (6), we are able to modify any existing many-valued logic, based on the complete distributive lattice of truth values, by weakening its negation and by using the intuitionistic implication (that is, the pseudocomplement of the complete distributive lattice of truth values in $\mathcal{B}$), in order to obtain a mZ paraconsistent logic.

Now we will apply this method to an important infinitary many valued logic: Zadeh's Fuzzy logic.

A. Da Costa paraconsistent Zadeh-fuzzy logic

Let us consider the original set of Zadeh fuzzy operators on the closed interval $[0, 1]$ of reals, with negation $\neg$, conjunction $\wedge$ and disjunction $\vee$, defined as follows:

1. $\neg x = 1 - x$;
2. $x \wedge y = min(x,y)$;
3. $x \vee y = max(x,y)$;

so that the conjunction and disjunction are the met and join operators of the complete distributive lattice $([0, 1], \leq)$. The closed interval of reals $[0, 1]$ is a total ordering, i.e., a complete distributive lattice, so that we can enrich the original fuzzy logic with intuitionistic implication (as in t-norm logics), that is with relative pseudocomplement.

Thus, for any $x, y \in [0, 1]$ we define

$$x \Rightarrow y \text{ by } \bigvee\{z | x \wedge z \leq y\} = max\{z | min(x,z) \leq y\}.$$

It is easy to verify that the fuzzy logic is not da Costa paraconsistent, because of the fact that $\neg 1 = 0$. In the standard fuzzy logic, the sentences are considered true if their truth value x satisfy $0 < \varepsilon_1 \leq x \leq 1$ for a prefixed value ϵ_1, so that the set of designated truth values is $D = [\varepsilon_1, 1]$ (each sentence with the truth value in D is considered as true sentence).

Consequently, as in the case of the 3-valued Kleene logic and 4-valued Belnap's bilattice logic, we need to change the original negation, where $\neg x = 1 - x$, in the way that $\neg 1 = \varepsilon_0 > 0$, where $\varepsilon_0 > 0$ is a prefixed positive infinitesimal value. In this way, with this very slightly changed fuzzy logic, we obtain a paraconsistent fuzzy mZ logic.

We recall that in any many-valued mZ logic the set of designated elements is exactly the singleton $\{1\}$ (this fact comes out from the Kripke-like semantics of the mZ system, where a sentence (proposition) is true only and only if it is true in *all* possible worlds in $\mathcal{W}$), thus the set of truth values in the paraconsistent fuzzy logic that is a member of mZ has to be changed from $[0, 1]$ for the set D of designated elements in the fuzzy logic, as follows:

Lemma 5: Let us consider a fuzzy logic with a matrix defined by a set of designated elements $D = [\varepsilon_1, 1]$, with $0.5 < \varepsilon_1 \leq 1$, and let us fix an infinitesimal positive value ε_0 such that $0 < \varepsilon_0 << \varepsilon_1$. Then we define the totally ordered lattice of logic truth values by $(\mathcal{B}, \leq, \wedge, \vee) = \{0\} \bigcup [\varepsilon_0, 1]$.

We define the intuitionistic implication $\Rightarrow$ by the relative-pseudocomplement in $\mathcal{B}$, and the negation operator for any $x \in \mathcal{B}$,

$$\neg x = \varepsilon_0 \vee (1 - x) = max(\varepsilon_0, 1 - x).$$

Then, the logic $(\mathcal{B}, \leq, \wedge, \vee, \Rightarrow, \neg)$ is a paraconsistent mZ fuzzy logic.

Proof: First of all let us show that $\neg$ is an antitonic additive operator, thus a split negation that can be modeled by Birkhof polarity and its incompatibility relation.

In fact, for any two $x, y \in \mathcal{B}$ if $x \geq y$ then $\neg x = max(\varepsilon_0, 1 - x) \leq max(\varepsilon_0, 1 - y) = \neg y$. Thus, it is antitonic. The antitonicity is preserved in the cases when $\neg x = 1$ as well. Let us show that it is also additive:

1. For the bottom element $0 \in \mathcal{B}$ we have that $\neg 0 = max(\varepsilon_0, 1) = 1$.
2. For $x, y \in \mathcal{B}$,

$$\neg(x \vee y) = \neg(max(x,y)) = max(\varepsilon_0, 1 - max(x,y))$$
$$= max(\varepsilon_0, min(1 - x, 1 - y)) = max(\varepsilon_0, (1 - x) \wedge (1 - y))$$
$$= \varepsilon_0 \vee ((1 - x) \wedge (1 - y)) = (\varepsilon_0 \vee (1 - x)) \wedge (\varepsilon_0 \vee (1 - x))$$
$$= \neg x \wedge \neg y.$$

Thus, $\neg$ is an additive antitonic operation, and, consequently, it is a split negation.

Let us define the hereditary incompatible relation for this split negation. The subset of join-irreducible elements of $\mathcal{B}$ is the complete distributive lattice $(\mathcal{W}, \sqsubseteq) = \widehat{\mathcal{B}} = [\varepsilon_0, 1]$, where we chose $\sqsubseteq$ to be inverse of $\leq$ in $\mathcal{B}$.

Consequently, the set of all hereditary subsets of the complete distributive lattice $\mathcal{W}$ is

$$\mathcal{H}_W = \{\downarrow_0^+ x \mid x \in \mathcal{W} \bigcup \{0\}\} = \{[\varepsilon_0, x] \mid x \in \mathcal{W} = [\varepsilon_0, 1]\},$$

where $[\varepsilon_0, \varepsilon_0] = \{\varepsilon_0\}$, so that there is the isomorphism $is : (\mathcal{H}_W, \subseteq) \rightarrow (\mathcal{B}, \leq)$ such that $is(\emptyset) = 0$ and for any $x \in \mathcal{W}$, $is(\downarrow_0^+ x) = x$. Thus, based on (6),

$$\mathcal{R} = \bigcup_{x \in [\varepsilon_0, 1]} [\varepsilon_0, x] \times [\varepsilon_0, max(\varepsilon_0, 1 - x)],$$

Consequently, for each $x \in \mathcal{W}$, $\lambda([\varepsilon_0, x]) = [\varepsilon_0, max(\varepsilon_0, 1 - x)]$, and it corresponds to $\neg x = max(\varepsilon_0, 1 - x)$.

Thus, this is a well defined paraconsistent fuzzy logic which belongs to the mZ logics.

□

It is easy to verify, that in the limit case for the set of designated elements $D = \{1\}$, that is when $\varepsilon_1 = 1$, we obtain that $\neg x \in D$ (that is, it is true) if and only if $x = 0$.

Notice that for each $max\{\varepsilon_0, 1-\varepsilon_1\} < x < \varepsilon_1$, we have that $\neg(x \wedge \neg x) \notin D$ (that is, it is not true) which satisfies the principle of non-contradiction Nd1 of da Costa negation weakening (consider, for example, the limit case when $\varepsilon_1 = 1$).

B. Paraconsistent subintuitionistic Gödel-Dummett logic

T-norm fuzzy logics are a family of non-classical logics [36], informally delimited by having a semantics which takes the real unit closed interval $[0, 1]$ for the system of truth values and functions called t-norms for permissible interpretations of conjunction.

Gödel-Dummett logic (the logic of the minimum t-norm) was implicit in Gödel's 1932 proof of infinite-valuedness of intuitionistic logic [37]. Later (1959) it was explicitly studied by Dummett who proved a completeness theorem for the logic [38].

Definition 9: Gödel-Dummett logics, $LC_n = (\mathcal{B}_n, \wedge, \vee, \Rightarrow, \neg, 0, 1)$, where for finite $n \geq 2$, $\mathcal{B}_n = \{0, \frac{1}{n-1}, ..., \frac{n-2}{n-1}, 1\}$ and for infinite n, $\mathcal{B}_\infty = [0, 1]$ (closed interval of reals) are both total orders, where $\wedge, \vee$ are the meet and join operators of these distributive complete lattices and $\Rightarrow, \neg$ are defined as in the IPC.
Thus, they are intermediate logics, that is,

IPC + $((A \Rightarrow B) \vee (B \Rightarrow A)$.

□

From the fact that this (added to IPC) axiom $(A \Rightarrow B) \vee (B \Rightarrow A)$ does not contain negation symbols, we conclude that the logic obtained by addition of this axiom to mZ axiom system would not change the paraconsistent weakening of the intuitionistic negation. Consequently, we are able to make the paraconsistent weakening of Gödel–Dummett logic in the same way as for IPC.

Lemma 6: The paraconsistent Gödel-Dummett logic is obtained by addition the axiom $(A \Rightarrow B) \vee (B \Rightarrow A)$ to mZ system . In this case we do not change $\mathcal{B}_n$ and we define $\mathcal{B}_\infty = \{0\} \bigcup [a, 1]$ for an enough big value $n >> 2$ and atom $a = \frac{1}{n-1}$, and the paraconsistent negation for any x in $\mathcal{B}_n$:

$\neg x = 1$ if $x = 0$; a otherwise.

Proof: It is easy to verify that $\neg$ is an antitonic operator. Let us show that it is an additive modal operator as well:

1. $\neg 0 = 1$.

2. Let us show that for each $x, y \in \mathcal{B}_n$,

$\neg(x \vee y) = \neg x \wedge \neg y$:

2.1 when $x = 0$ (or $y = 0$), we have

$\neg(0 \vee y) = \neg y = 1 \wedge \neg y = \neg 0 \wedge \neg y$.

2.2 when both x and y are different from 0:

$\neg(x \vee y) = a = a \wedge a = \neg x \wedge \neg y$.

□

Notice that $\mathcal{W} = \widehat{\mathcal{B}}_n = \mathcal{B}_n \backslash \{0\}$, so that the incompatibility relation is based on (6)

$\mathcal{R} = \bigcup_{x \in \mathcal{W}} (\downarrow_0^+ x) \times (\downarrow_0^+ \neg x)) = \bigcup_{x \in \mathcal{W}} (\downarrow_0^+ x) \times \{\frac{1}{n-1}\}$,

so that $\lambda(\downarrow_0^+ x) = \{\frac{1}{n-1}\}$ for each $x \in \mathcal{W}$.

It is easy to verify that $\neg_I \preceq \neg$, as demonstrated by Corollary 3 for mZ logic, and $\neg_I \neq \neg$. It is interesting to note that

$\neg x \neq \neg_I x =_{def} (x \Rightarrow a) = 1$ if $x \in \{0, a\}$; a otherwise.

Both formulae, $A \vee \neg A$ and $\neg(A \wedge \neg A)$, are not theorems. In fact for each $x \in \mathcal{B}_n$ such that $x \neq 0$, we have that

$\neg(x \wedge \neg x) = a < 1$,

and for every $x \notin \{0, 1\}, \neg x \vee x = x < 1$.

Notice that by the paraconsistent weakening of the minimal cardinality Gödel-Dummett logic (when $n = 3$), we obtain the paraconsistent Kleene logic in Lemma 2.

VII. Conclusion

In this paper we analyzed the principle of constructive negation and we have shown that the a paraconsistent split negation used in mZ system is constructive as intuitionistic negation (which is a special case of a split but non paraconsistent negation). The main theoretical and philosophical results are obtained by demonstration that mZ system is a *subintuitionistic* constructive paraconsistent logic and then we presented the strict relationship between intuitionistic and new paraconsistent negation in mZ.

The significant part of this paper is dedicated to a number of applications of these subintuitionistic paraconsistent mZ logics. In particular, we dedicated an attention to Logic Programming with Fitting's fixpoint semantics (w.r.t. the knowledge ordering) by using mZ logics with minimal cardinalities, obtained by paraconsistent da Costa's weakening of the 3-valued Kleene and 4-valued bilattice-based Belnap's logics.

Moreover, we defined the canonical constructions of infinite-valued mZ logics and paraconsistent weakening of the classic (Zadeh) fuzzy logic and of the Gödel-Dummet t-norm intermediate logic.

The Kripke-style semantics for these paraconsistent negations are defined as *modal* additive negations: they are a conservative extension of the positive fragment of Kripke semantics for intuitionistic propositional logic [1], where only the satisfaction for negation operator is changed by adopting an incompatibility accessibility relation for this modal operator which comes from Birkhoff polarity theory based on a Galois connection for negation operator.

Future work: With this paper I finished my research, published in last decade [1], [39], [40], [2], in weakening of negation of paraconsistent Da Costa system.

However, this work opens another interesting directions of research for this subintuitionistc logic. For example, it is well known that Cartesian Closed Categories (CCC) are models for the typed lambda-calculus. By enriching CCCs with finite coproducts (for logic disjunction) and by classifier subobjects, we obtain the topoi as models for the intuitionistic logic (or, equivalently, Heyting algebras). From the algebraic point of view, mZ logic is the extension of the Heyting algebra $((\mathcal{B}, \leq), \wedge, \vee, \Rightarrow)$ with a paraconsistent modal negation $\neg$. Thus, another interesting investigation may be to consider which kind of enrichment of topos is needed in order to obtain a categorial model for this system of paraconsistent but still constructive logics mZ.

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