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Differentiation Functors and Category Interpretation of Optimality Conditions

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Abstract—Operator derivatives are determined as functors. Necessary optimality conditions with category interpretation are proved for abstract optimization control problems. Finite dimensional extremum problems and an optimization control problem for nonlinear parabolic equation with state constraints are considered as examples.

Keywords—Category, control, differentiation functor, operator derivative, optimality conditions.

I. Introduction

OPTIMIZATION control problems are solved by means of necessary conditions of optimality frequently. First order optimality conditions used the differentiation of the state functional (see, for example, the stationary condition, Euler equation, the variational inequality, the maximum principle, etc.). So the optimization control theory can be interpreted as an application of the differentiation theory.

The differentiation is an operation of the local linearization [1]. The nonlinear phenomenon has become weakly apparent in a small enough set. Hence the regular enough nonlinear object can be approximated by a linear one. For example, the smooth curve can be approximated in a neighbourhood of a point by its tangent in this point. However the definition of the tangent uses the derivative of the function. We apply the differentiation whenever a nonlinear object is analyzed by means of its linear approximation.

Note that the differentiation relates with the local structure of the object only. If functions (functionals, operators) are equal in a neighbourhood of a point, then it has the same derivatives in this point. So the derivative characterizes the local properties of the class of objects, but not a concrete object. These objects are equivalent in some way. This equivalence class is the germ of functions (functionals, operators) in this point [2]. Therefore there exists a natural relation between the differentiation and germs theory.

The differentiation transforms the germ of operators to a linear operator, which is its derivative in the given point. This map can be interpreted as a functor. It transforms the category, which has germs of operators as morphisms, to the category, which has linear operators as morphisms. So we can apply the categories theory [3] for the interpretation of the differentiation.

The differentiation functor was defined in [4] without the germs theory. These results were used for the analysis of unconditional extremum problems there. The definition of the differentiation functor with using germs theory and its application to the extremum theory by means of the inverse function theorem where considered in [5]. We will define partial differentiation functors. Necessary optimality conditions with category interpretation will be proved for abstract optimization control problems with using implicit function theorem. Finite dimensional extremum problems and an optimization control problem for nonlinear elliptic equation with state constraints will be considered as examples.

II. Differentiation Functor and its Application to the Extremum Theory

We consider the set of pairs (X, x), where X is a Banach space, and x is a fixed point of X. For all pairs (X, x) and (Y, y) determine an operator $L: X \to Y$ that is Frechet differentiable at the point x such that $Lx = y$. Two operators are equivalent if they coincide at a neighbourhood of the point x. The relevant equivalence class, namely the germ of the operator L at the point x, is denoted by L_x. We determine the category Γ with Banach spaces with fixed points as the objects and the germs of differentiable operators as the morphisms.

We now define a map D from Γ to the category B of Banach spaces with linear continuous operators. For all object (X, x) and the morphism L_x of the category Γ with the beginning (X, x) and the end (Y, y) we determine

$$D(X, x) = X,\ DL_x = L'(x).$$

This map is a functor. It is called the *differentiation* [4]; and the value $D\psi$ at the germ ψ is called the *derivative of the morphism* ψ of the category Γ [5].

Determine a category Σ with Banach spaces with fixed points as the objects. Consider the germs of operators that are continuously differentiable at a neighbourhood of the fixed point and have invertible derivative at this point. Let it be the morphisms of Σ. Then Σ is the subcategory Σ of Γ; besides its morphisms are isomorphisms.

There exists an application of these notions to the extremum theory. Let $A: Y \to V$ be a state operator, where *V* and *Y* are Banach spaces. The state of a system is described by the equation $Ay = v,$ where *v* is a control, and *y* is the state function. Suppose for all control *v* of *V* this equation has a unique solution $y = Lv$ from the space *Y*. Determine the state functional $I: V \to \mathbb{R}$ by the equality $I(v) = J(v) + K(Lv),$ where $J: V \to \mathbb{R},\ K: Y \to \mathbb{R}$ are given functional. We have the problem of the minimization of the functional *I* on the space *V*.

If *v* is a point of the local minimum of the functional *I* on the space *V*, J_v and K_y are the morphisms of Γ, and A_y is the morphism of Σ, where $y = Lv$, then

$$DJ_v + \left[H(DA_y)\right]^{-1} DK_y = 0,$$

where *H* is the general cofunctor from B to the sets category that is determined by the object $\mathbb{R}$ [4]. This result was extended to the problem of the minimization of a functional on the convex set [5]. However it was an optimization problem, where the control is an absolute term of the state equation only. Besides the state functional was the sum of the functional *J* and *K* there. We will consider the general case of the state equation and the functional. So we will determine partial differentiation functors.

III. Partial Differentiation Functors and Abstract Optimization Control Problem

Consider a continuously differentiable operator $A: V \times Y \to Z$ and a functional $I: V \times Y \to \mathbb{R}$, where *V,Y,Z* are Banach spaces. Suppose for all control $v \in V$ there exists a unique state $y = Lv$ from *Y* such that $A(v, y) = 0.$ We have the problem of the minimization for the functional $v \to I(v, Lv)$ on the space *V*.

Let $(V, v),\ (Y, y),\ (V \times Y, w)$ be objects of Γ, where $w = (v, y).$ Then A_w is the morphism of Γ. Its derivative DA_w is the pair $\left(A_v(w), A_y(w)\right),$ where $A_v(w)$ is the derivative of the map $v \to A(v, y)$ at the point *v*, and $A_y(w)$ is the derivative of $y \to A(v, y)$ at *y*. Denote by $V \oplus Y$ the coproduct of the objects of the category B. Its morphisms $A_v(w): V \to Z,\ A_y(w): Y \to Z$ determine a cocone. Then $\iota_V DA_w = A_v(w),\ \iota_Y DA_w = A_y(w),$ where ι_V and ι_Y are the canonic inclusions of *V* and *Y* to $V \oplus Y$.

Suppose the beginning (W, w) of the morphism A_w of the category Γ is the coproduct $(V, v) \oplus (Y, y).$ The values $\iota_V DA_w$ and $\iota_Y DA_w$ are called the *partial derivatives* $D_V A_w$ and $D_Y A_w$ of A_w. We have the equalities $DA_w = D_V A_w \oplus D_Y A_w$ and $DI_w = D_V I_w \oplus D_Y I_w$. Consider the pair $F = (I, A)$ and the matrix $F'(w)$ of its partial derivatives at the point *w*. It is the derivative DF_w of the morphism F_w of Γ. But it is the morphism of the category B with beginning $V \oplus Y$ and $\mathbb{R} \otimes Z,$ that is the product of the objects of B.

Return to our optimization control problem. It can be transformed to the problem of the minimization for the smooth functional $S = IQ$ on the space *V*, where $Q = \left(E, L\right),$ and *E* is the unit operator on *V*.

Theorem 1. Suppose *v* is a point of local minimum of the functional *S* on the space *V*, F_w is the morphism of Γ, and C_{Lv} is the morphism of Σ, where $w = (v, Lv),\ \ Cy = A(v, y).$ Then

$$D(F_w Q_v) = 0. \qquad (1)$$

Proof. The derivative $A_y(w)$ is invertible. So the operator *L* is differentiable at the point *v* because of the implicit function theorem. Then L_v and Q_v are morphisms of the category Γ. Therefore the functional *S* is differentiable; and necessary condition of local extremum $S'(v) = 0$ is true. It can be transformed to the equality $DS_v = 0.$ Then the operator $R = AQ$ is differentiable too. Using state equation, from the equality $R(v + \sigma h) - Rv = 0$ for all number σ and $h \in V$, we get $R'(v) = 0;$ so $DR_v = 0.$ Then we obtain

$$F_w Q_v = \left(I_w, A_w\right) Q_v = \left(I_w Q_v, A_w Q_v\right) = \left(S_v, R_v\right)$$

because of the definition of the morphism F_w. So the equality (1) is true.

We give some corollaries of Theorem 1.

Corollary 1. Under the conditions of Theorem 1 we have the equality $I_v(v, y) = \left[A_v(v, y)\right]^* p,$ where *y* is the solution of the state equation $A(v, y) = 0,$ and *p* is the solution of the adjoint equation $\left[A_y(v, y)\right]^* p = I_y(v, y).$

Corollary 2. Let $x = (x_0, x_1, ..., x_n)$ be a point of the local extremum of the function $f_0 = f_0(x)$ under the equalities $f_i(x) = 0,\ i = 1, 2, ..., n;$ and all functions are continuously differentiable at a neighbourhood of the point *x*. Then its Jacobian, that is determined by the functions $f_0, f_1, ..., f_n$ at the point *x*, is equal to zero.

Consider the minimization problem for the function $f_0 = f_0(x)$ under the equalities $f_i(x) = 0,\ i = 1, 2, ..., n,$ where $x = (x_1, ..., x_{n+r}).$ Suppose all functions are smooth enough. Fixe a vector $\alpha = (\alpha_1, ..., \alpha_n),$ where $\alpha_i \in \{1, ..., n + r\},\ \alpha_i \neq \alpha_j\ \forall i \neq j$. Determine the matrixes

$$F_{\beta_i}^{\alpha}(x) = \begin{pmatrix} \partial_{\alpha_1} f_0(x) & ... & \partial_{\alpha_n} f_0(x) & \partial_{\beta_i} f_0(x) \\ \partial_{\alpha_1} f_1(x) & ... & \partial_{\alpha_n} f_1(x) & \partial_{\beta_i} f_1(x) \\ ... & ... & ... & ... \\ \partial_{\alpha_1} f_n(x) & ... & \partial_{\alpha_n} f_n(x) & \partial_{\beta_i} f_n(x) \end{pmatrix},\ i = 1, 2, ..., r,$$

where $\beta_i \in \{1,...,n+r\}$, $\beta_i \neq \beta_j$ for all $i \neq j$ and $\beta_i \neq \alpha_j$ for all *i,j*; $\partial_k f_m(x)$ is the derivative of the function f_m with respect to x_k at the point *x*.

Corollary 3. If *x* is a point of the local extremum of the function f_0 under the given constraints, then $\left|F_{\beta_i}^{\alpha}(x)\right| = 0,\ i = 1,2,...,r.$

We determine necessary conditions of the extremum without Lagrange multipliers. However these results can be transformed to the standard form.

IV. Optimization Control Problems with Constraints

Consider again a continuously differentiable operator $A: V \times Y \to Z$ and a functional $I: V \times Y \to \mathbb{R}$, where *V,Y,Z* are Banach spaces. Let *U* be a convex closed subset of *V*. Suppose for all control $v \in U$ there exists a unique state $y = Lv$ from *Y* such that $A(v,y) = 0.$ We have the problem of the minimization for the functional $v \to I(v, Lv)$ on the set *U*.

Theorem 2. Under the conditions of Theorem 1 suppose *v* is a point of a local minimum for the functional $v \to I(v, Lv)$ on the set *U*. Then *v* satisfies the variational inequalities

$$\pi_i D\left(F_w Q_v\right)(u-v) \geq 0\ \forall u \in U_i,\ i = 1,2, \tag{2}$$

where $U_1 = U,\ U_2 = V,\ \pi_1$ and π_2 are canonic projections.

Proof. Let *v* be a point of a local minimum for the functional $v \to I(v, Lv)$ on the set *U*. Then we have the inequality

$$I(\theta, L\theta) - I(v, Lv) \geq 0\ \forall \theta \in O,$$

where the subset *O* of *U* is a neighbourhood of the point *v*. Let $u \in U$ be fixed. Chose a positive number σ such that the inclusion $v + \sigma(u-v) \in O$ is true. Determine the control $\theta = v + \sigma(u-v).$ We get

$$I\left(v + \sigma(u-v), L\left[v + \sigma(u-v)\right]\right) - I(v, Lv) \geq 0. \tag{3}$$

Let Ξ maps the morphism $\psi = L_x$ of Γ to the value *Lx*. So the derivative of the morphism satisfies the equality

$$\Xi L_{x+h} = \Xi L_x + DL_x h + \eta(h),$$

where $\|\eta(h)\| = o\left(\|h\|\right).$ Then we have

$$\Xi I_{w+h} = \Xi I_w + D_V I_w h_V + D_Y I_w h_Y + \eta_I(h),$$

$$\Xi A_{w+h} = \Xi A_w + D_V A_w h_V + D_Y A_w h_Y + \eta_A(h),$$

where $h = (h_V, h_Y),\ \|\eta_I(h)\| = o\left(\|h\|\right),\ \|\eta_A(h)\| = o\left(\|h\|\right).$

Determine

$$w = (v, Lv),\ h = \left(\sigma(u-v), h_Y\right),\ h_Y = L\left[v + \sigma(u-v)\right] - Lv.$$

Using implicit function theorem, we get

$$h_Y = \sigma L'(v)(u-v) + \eta(\sigma) = \sigma DL_v(u-v) + \eta(\sigma),$$

where $\|\eta(\sigma)\| = o(\sigma).$ Then we obtain

$$\Xi I_{w+h} = \Xi I_w + \sigma D_V I_w(u-v) + \sigma D_Y I_w DL_v(u-v) + \eta_1(\sigma),$$

where $\|\eta_1(\sigma)\| = o(\sigma).$ Devise the inequality (3) by σ and pass to the limit as $\sigma \to 0.$ We have

$$D_V I_w(u-v) + D_Y I_w DL_v(u-v) \geq 0\ \forall u \in U.$$

It can be transform to

$$DS_v(u-v) \geq 0\ \forall u \in U.$$

We have also

$$A\left(v + \sigma g, L\left(v + \sigma g\right)\right) - A(v, Lv) = 0\ \forall g \in V.$$

So we get

$$D_V A_w g + D_Y A_w DL_u g = 0\ \forall g \in V.$$

It is equivalent to the inequality

$$D_V A_w(u-v) + D_Y A_w DL_u(u-v) \geq 0\ \forall u \in V.$$

Then

$$DR_v(u-v) \geq 0\ \forall u \in V.$$

We have the equalities $\pi_1\left(F_w Q_v\right) = S_v,\ \pi_2\left(F_w Q_v\right) = R_v.$ Using last inequalities, we get (2).

If $U = V,$ then the variational inequality (2) can be transformed to the equality (1).

Corollary 4. Under the conditions of Theorem 2 we have the variational inequality

$$\left\langle I_v(v,y) = \left[A_v(v,y)\right]^* p, u - v\right\rangle \geq 0\ \forall u \in U, \tag{4}$$

where $\langle \lambda, \mu \rangle$ is the value of the linear continuous functional λ at the point μ, *y* is the solution of the state equation $A(v,y) = 0,$ and *p* is the solution of the adjoint equation

$$\left[A_y(v,y)\right]^* p = I_y(v,y). \tag{5}$$

The state equation has a unique solution for our case. So there exists a bijection between the set of controls and the set of the state functions. So we have equivalence between controls and states. The single pair "control-state" was considered for solving systems described by singular systems [6,7]. We consider other case as an example. Let us have an optimization problem for a system with state constraints only. There exist difficulties for solving this problem by means of standard methods because we do not know how we can variate the control for saving state constraints. However we can rearrange the control and the state function. So we will use our results with state variation.

Consider an example. Let Ω be an open bounded *n*-dimensional set. We have the equation

$$z' - \Delta z + |z|^{\rho} z = g \tag{6}$$

in the set $Q = \Omega \times (0,T),$ where z' is the derivative of *z* with respect to *t*, ρ is a positive constant for $n = 2,$ and $0 < \rho \leq 2/(n-2)$ for $n > 2.$ For all *g* from $Y = L_2(Q)$ this equation has a unique solution $z = Mg$ from the space

$$\left\{ z \,\middle|\, z \in L_\infty\left(0,T; H_0^1(\Omega) \cap L_{\rho+2}(\Omega)\right),\ z' \in L_2(Q),\ z|_{t=0} = 0 \right\},$$

besides the operator M is $*$-weakly continuous (see [8], ch. VI, Theorem 1.1). We have $H_0^1(\Omega) \subset L_{2(\rho+1)}(\Omega)$ because of Sobolev theorem. So we have $\Delta z \in L_2(Q)$. Then the solution of our boundary problem is the point of the space

$$V = \left\{ z \middle| \, z \in L_\infty\left(0,T;H_0^1(\Omega)\right), z' \in L_2(Q), \Delta z \in L_2(Q), z\big|_{t=0} = 0 \right\}.$$

Let U be a convex closed subset of the space V. Consider the functional

$$I(z,g) = \frac{1}{2}\left\|z-\zeta\right\|^2_{L_2\left(0,T;H_0^1(\Omega)\right)} + \frac{\chi}{2}\left\|g\right\|^2_{L_2(Q)},$$

where $\zeta \in L_2\left(0,T;H_0^1(\Omega)\right)$ is a given function, $\chi > 0$, and the functions z and g satisfy the given equation. We have the problem of the minimization for the functional I under the condition $z \in U$. Using standard technique (see, for example, [9]) we prove the solvability of this problem.

Necessary conditions of optimality for nonlinear parabolic equations with state constraints are well known. It is systems with fixed final time [10-12], optimization problems with finite quantity of integral equalities and inequalities [13], pointwise constraints [14-16], and time optimal problem [17]. There exist a few results for the general state constraints. It uses regularization method [14] or Ekeland principle [18]. We will obtain standard variational inequality as necessary conditions of optimality by means of Corollary 4.

Theorem 3. The optimal control is determined by the formula

$$g = -\chi^{-1} p, \tag{7}$$

where the function p is equal to zero on the boundary of the given set and for the final time; it satisfies the conditions

$$z' - \Delta z + |z|^\rho z = -\chi^{-1} p, \tag{8}$$

$$-p' - \Delta p + (\rho+1)\left|z\right|^\rho p = q, \tag{9}$$

$$\int_Q \left(\Delta\zeta - \Delta z - q\right)(u - z)dQ \ge 0 \;\; \forall u \in U. \tag{10}$$

Proof. We use Corollary 4. Let the state function z of our system be a “control” v, and the control g be a “state function” y of the general problem. Determine the operator A by the formula $A(v,y) = v' - \Delta v + \left|v\right|^\rho v - y$. Then the operator $Lv = v' - \Delta v + |v|^\rho v$ is differentiable. So we can apply Theorem 2.

We have

$$\left\langle I_v(v,y), h\right\rangle = \int_Q \left(\nabla v - \nabla\zeta\right)\nabla h dQ = \int_Q \left(\Delta\zeta - \Delta v\right) h dQ,$$

$$\left\langle \left[A_v(v,y)\right]^* p, h\right\rangle = \left\langle p, \left[A_v(v,y)\right]h\right\rangle = \int_Q \left(h' - \Delta h + (\rho+1)\left|v\right|^\rho h\right) p dQ = \int_Q \left(-p' - \Delta p + (\rho+1)\left|v\right|^\rho p\right) h dQ$$

for all $h \in V$ and for all smooth functions p that is equal to zero on the boundary of the given set and for the final time. Then the variational inequality (4) can be transformed to

$$\int_Q \left[\left(\Delta\zeta - \Delta v\right) - \left(-p' - \Delta p + (\rho+1)\left|v\right|^\rho p\right)\right](u - v)dQ \ge 0$$

for all $u \in U$. We have the equalities $I_y(v,y) = \chi y$, $\left[A_y(v,y)\right]^* p = -p$. Then we transform the adjoint equation (5) to $y = -\chi^{-1} p$. So the equality (7) is true. Besides we get (8) because of the state equation (6). Determine the function q as the right side of the equality (9). Then the last variational inequality can be transformed to (10). This completes the proof of Theorem 3.

We obtained necessary conditions of optimality in the standard form. It can be solved by means of iterative methods. However this system has peculiarities [7]. The state equation is applied for finding the state function as a rule, the adjoint function is determined from the adjoint equation, and the variational inequality is used for finding the control. But we have another algorithm. If the function q is known for the fixed iteration, then we find the state function z from the variational inequality (10). Then we determine the functions p and q from the equalities (8) and (9). Hence we do not solve the state equation and the adjoint one. It is interpreted as the formulas for finding the functions p and q. The control is determined by the formula (7). The general difficulty for the fixed iteration is solving of the variational inequality (10). However it is known numerical methods for finding the solutions of variational inequalities [19]. The analogical results can be obtained for other optimization control problems with state constraints, for example for elliptic equations [20].

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