INTERNATIONAL JOURNAL OF PURE MATHEMATICS
DOI: 10.46300/91019.2022.9.2 Volume 9, 2022

# Derivation of a numerical method with free second-order derivatives

Young Hee Geum
Department of Applied Mathematics, Dankook University, Cheonan, Korea

**Abstract—We have proposed the second-derivative-free numerical method and determined the control parameters to converge cubically. In addition, we have developed the order of convergence and the asymptotic error constant. Applying this iterative scheme to a variety of examples, numerical results have shown a successful asymptotic error constants with cubic convergence.**

**Keywords—second-derivative-free, order of convergence, asymptotic error constant, iterative method , multiple root, root-finding**

## 1.. INTRODUCTION

MANY researchers[4,5,6,7,8] have been iterested in developping the iteration methods and deriving the asymptotic error constant to find the roots of nonlinear equations. The Newton's method is one of the most well-known iteration method and is applied.

Suppose that a function $f:\mathbb{C}\to\mathbb{C}$ has a multiple zero $\alpha$ with integer multiplicity $m\ge 1$ and is analytic[1,2,3] in a small neighborhood of $\alpha$. We find an approximated $\alpha$ by a scheme

$$x_{n+1}=g(x_n),\ n=0,\ 1,\ 2,\cdots, \tag{1}$$

where $g:\mathbb{C}\to\mathbb{C}$ is an iteration function and $x_0\in\mathbb{C}$ is given. Then we find an approximated $\alpha$ using an iterative method. To solve the equation, we develoop the following scheme:

$$g(x)=x-\lambda f(x-\mu h(x))/f'(x) \tag{2}$$

where

$$h(x)=\begin{cases} f(x)/f'(x), & if\ x=\alpha \\ \lim_{x\to\alpha} f(x)/f'(x), & if\ x=\alpha. \end{cases} \tag{3}$$

Let $p\in \boldsymbol{N}$ be given and $g(x)$ satisfy the following relation

$$\begin{cases} \left|\frac{d^p}{dx^p}g(x)\right|_{x=\alpha}=|g^{(p)}(\alpha)|<1, & \text{if } p=1. \\ g^{(i)}(\alpha)=0 \text{ for } 1\le i\le p-1 \text{ and } g^{(p)}(\alpha)=0, & \text{if } p\ge 2. \end{cases} \tag{4}$$

Since $g(x)$ is continuous at $x=\alpha$, $g(x)$ is represented by

$$g(x)=\begin{cases} x-\lambda F(x), & if\ x=\alpha \\ x-\lambda\lim_{x\to\alpha}F(x), & if\ x=\alpha. \end{cases} \tag{5}$$

where $z(x)=x-\mu h(x)$ and $F(x)=\frac{f(x-\mu h(x))}{f'(x)}$.

By Corollary 1 and Corollary 2, we have $[f(z)]^{(k)}_{x=\alpha}=0,\ 0\le k\le m-1$ and $f(\alpha)=f'(\alpha)=\cdots=f^{(m-1)}(\alpha)=0,\ f^{(m)}\neq 0$. Using L'Hospital's rule repeatedly, we obtain

$$\lim_{x\to\alpha}F(x)=\frac{[f(z)]^{(m-1)}_{x=\alpha}}{[f'(x)]^{(m-1)}}=0 \tag{6}$$

The next corollary is useful to calculate $g'(\alpha)$, $g''(\alpha)$ and $g'''(\alpha)$.

*Corollary 1:* Suppose $f:\mathbb{C}\to\mathbb{C}$ has a multiple root $\alpha$ with a given integer multiplicity $m\ge 1$ and is analytic in a small neighborhood of $\alpha$. Then the function $h(x)$ and its derivatives up to order 3 evaluated at $\alpha$ has the following properties with $\theta_j=\frac{f^{(m+j)}(\alpha)}{f^{(m)}(\alpha)}, j\in\mathbb{N}$:

(i) $h(\alpha)=0$

(ii) $h'(\alpha)=\frac{1}{m}$

(iii) $h''(\alpha)=-\frac{2}{m^2(m+1)}\theta_1$

(iv) $h^{(3)}(\alpha)=\frac{6}{m^3(m+1)}\left\{{\theta_1}^2-\frac{2m}{m+2}\theta_2\right\}$

*Corollary 2:* Let $f$ stated in Corollary 1 have a multiple root $\alpha$ with a given multiplicity $m\ge 1$. Let $z(x)=x-\mu h(x)$ and $h(x)$ be defined by Eq.(3). Then the following hold:

$$\frac{d^k}{dx^k}f(z)\Big|_{x=\alpha}=[f(z)]^{(k)}|_{x=\alpha}$$

$$=\begin{cases} 0, & if\ 0\le k\le m-1 \\ f^{(m)}(\alpha)\cdot z'(\alpha)^m, & if\ k=m \\ f^{(m+1)}(\alpha)\cdot z'(\alpha)^{m+1}+f^{(m)}(\alpha)\frac{m(m+1)}{2}\cdot z'(\alpha)^{m-1}z''(\alpha), & if\ k=m+1 \\ f^{(m+2)}(\alpha)\cdot z'(\alpha)^{m+2}+f^{(m+1)}(\alpha)\frac{(m+1)(m+2)}{2}\cdot z'(\alpha)^m z''(\alpha) +f^{(m)}(\alpha)\cdot L_{m+2}(\alpha), & if\ k=m+2 \end{cases}$$

where $L_k=\binom{k}{3}t^{k-4}\{t\cdot(-\mu h''')+\frac{3}{4}(k-3)\mu^2h''(\alpha)^2\}$.

## 2.. CONVERGENCE ANALYSIS

We establish some relationships between $\lambda, m, g'(\alpha)$, $g''(\alpha)$ and $g'''(\alpha)$, for maximum order of convergence[9,10,11].

We rewrite Eq.(2) into

$$(g-x)\cdot f'(x)=-\lambda f(z). \tag{7}$$

where $f=f(x), f'=f'(x)$ and $z=x-\mu h(x)$ are used for concise and the symbol $'$ denotes the derivative with respect to $x$.

Differentiating both sides of Eq(7) with respect to $x$, we obtain

$$(g'-1)\cdot f'+(g-x)\cdot f''(x)=-\lambda[f(z)]^{(1)} \tag{8}$$

Since $g'$ is continuous at $\alpha$, we have

$$g'(x)-1=\begin{cases} F_1(x), & if\ x\neq\alpha \\ \lim_{x\to\alpha}F_1(x), & if\ x=\alpha, \end{cases} \qquad (9)$$

where $F_1(x)=-(g-x)f''(x)-\lambda[f(z)]^{(1)}/f'$.
Using Corollary 2 and $g(\alpha)=\alpha$, we have the following:

$$(g-x)f''(x)]^{(k)}_{x=\alpha}$$

$$=\begin{cases} 0, & if\ 0\leq k\leq m-2, m\geq 2 \\ (m-1)(g'-1)f^{(m)}(\alpha), & if\ k=m-1, \end{cases} \qquad (10)$$

$$[f(z)]^{(1)}\Bigg]^{(k)}_{x=\alpha}=\begin{cases} 0, & if\ 0\leq k\leq m-2,\ m\geq 2 \\ f^{(m)}(\alpha)(1-\frac{\mu}{m})^m, & if\ k=m-1, \end{cases} \qquad (11)$$

Substituting Eq.(10) and Eq.(11) into Eq.(9), we have

$$g'(\alpha)-1=-(m-1)(g'(\alpha)-1)-\lambda(1-\frac{\mu}{m})^m$$

To obtain $g'(\alpha)=0$, we get

$$m=\lambda\left(1-\frac{\mu}{m}\right)^m=\lambda t^m \qquad (12)$$

where $t^m=1-\frac{\mu}{m}$.
Differentiate both sides of Eq(8) with respect to $x$, we get

$$g''+2(g\prime-1)\cdot f''+(g-x)\cdot f^{(3)}=-\lambda[f(z)]^{(2)} \qquad (13)$$

We rewrite

$$g''(x)=\begin{cases} F_2(x), & if\ x\neq\alpha \\ \lim_{x\to\alpha}F_2(x), & if\ x=\alpha, \end{cases} \qquad (14)$$

where

$$F_2(x)=-2(g'-1)\cdot f''-(g-x)\cdot f^{(3)}-\lambda[f(z)]^{(2)}/f'.$$

We can get the numerator of $F_2(x)$ by computition similar to that done in $F_1(x)$

$$-2(g'-1)f''-(g-x)f^{(3)}-\lambda[f(z)]^{(2)}$$

$$=\begin{cases} 0, & if\ 0\leq k\leq m-3 \\ f^{(m)}(\alpha)(m-\lambda t^m), & if\ k=m-2 \\ f^{(m+1)}(\alpha)[(m+1)-\lambda(t^{m+1}-t^m+t^{m-1})] & \\ -g''f^{(m)}(\alpha)\frac{(m+2)(m-1)}{2} & if\ k=m-1, \end{cases} \qquad (15)$$

From Eq.(14) and Eq.(15), we get

$$g''=\frac{2\theta_1}{m(m+1)}\{(m+1)-\lambda(t^{m+1}-t^m+t^{m-1})\} \qquad (16)$$

From Eq.(16), to have $g''(\alpha)=0$ we get the following relation,

$$m+1=\lambda(t^{m+1}-t^m+t^{m-1}) \qquad (17)$$

Differentiate both sides of Eq.(13) with respect to $x$ to obtain

$$g^{(3)}\cdot f'+3g''\cdot f''+3(g'-1)\cdot f^{(3)}+(g-x)\cdot f^{(4)}=-\lambda[f(z)]^{(3)}. \qquad (18)$$

We rewrite

$$g^{(3)}(x)=\begin{cases} F_3(x), & if\quad x\neq\alpha \\ \lim_{x\to\alpha}F_3(x), & if\quad x=\alpha, \end{cases} \qquad (19)$$

where

$$F_3(x)=-3g''f''-3(g'-1)f^{(3)}-(g-x)f^{(4)}-\lambda[f(z)]^{(3)}/f'$$

Hence, we have

$$-3g''f'']^{(k)}_{x=\alpha}-3(g'-1)f^{(3)}]^{(k)}_{x=\alpha}$$
$$-(g-x)f^{(4)}]^{(k)}_{x=\alpha}-\lambda[f(z)]^{(3)}]^{(k)}_{x=\alpha}$$

$$=\begin{cases} 0, & if\ 0\leq k\leq m-4 \\ f^{(m)}(\alpha)(m-\lambda t^m), & if\ k=m-3 \\ f^{(m+1)}(\alpha)\{m+1-\lambda(t^{m+1}-t^m+t^{m-1})\}, & if\ k=m-2 \\ -\frac{(m-1)(m^2+4m+6)}{6}g^{(3)}f^{(3)}+f^{(m+2)}(\alpha)(m+2) & \\ -\lambda\{f^{(m+2)}(\alpha)t^{m+2}-f^{(m+1)}(\alpha)\theta_1\frac{m+2}{m}(t^m-t^{m+1}) & \\ -f^{(m)}(\alpha)L_{m+2}(\alpha)\}, & if\ k=m-1, \end{cases} \qquad (20)$$

Consequently, we have

$$g^{(3)}(\alpha)=\frac{6}{m(m+1)(m+2)}$$

$$\left[\theta_2(m+2)-\lambda\{\theta_2 t^{m+2}+{\theta_1}^2(t^m-t^{m+1})\frac{m+2}{m}+L_{m+2}(\alpha)\}\right]. \qquad (21)$$

where $L_k=\binom{k}{3}t^{k-4}\{t\cdot(-\mu h''(\alpha))+\frac{3}{4}(k-3)\mu^2 h''(\alpha)^2\}$

*Theorem 1:* Let $f:\mathbb{C}\to\mathbb{C}$ have a zero $\alpha$ with integer multiplicity $m\geq 1$ and be analytic in a small neighborhood of $\alpha$. Let $\theta_1,\theta_2$ be defined as in Corollary 1. Let $t$ be a root of $\rho(t)$ defined in (20). Let $x_0$ be an initial guess chosen in a sufficiently small neighborhood of $\alpha$. Then iteration method (2) with $\mu=m(1-t)$ has order 3 and its asymptotic error constant $\eta$ as follows:

$$\eta=\frac{1}{6}|g^{(3)}(\alpha)|=\frac{1}{m(m+1)(m+2)}|\phi_1\theta_1^2+\phi_2\theta_2|,$$

where $\phi_1=-t^{m-2}\lambda q_1(t), \phi_2=m+2-\lambda t^{m-2}q_2(t)$, $q_1(t)=-\frac{(m+2)(t-1)^2\{2(m+1)t-m+1\}}{2m(m+1)}$ and $q_2(t)=t(t^3-2t+2)$.

## 3.. Numerical Results

In these experiments, we choose 300 as the minimum number of digits of precision by assigning *$MinPrecision=250* in Mathematica to obtain the specified nominal accuracy. We set the error bound $\epsilon$ to $0.5\times 10^{-235}$ for $\mid x_n-\alpha\mid<\epsilon$ and evaluate the $n^{th}$ order derivative of the complicated nonlinear functions using the Mathematica[12] command ***D***[f, $\{x,n\}$].

As an example for the convergence, we investigate the order of convergence and the asymptotic error constant with a function $f(x)=\{x^{10}-\sqrt{3}x^3\cos(\pi x/6)+1/(x^2+1)\}(x-1)$ having a real zero $\alpha=1.0$ of multiplicity 2. We choose $x_0=0.92$ as an initial guess. Table 1 verifies cubic convergence. The computed asymptotic error constants are in sucessful agreement with theoretical asymptotic error constants $\eta$ up to 10 significant digits. The computed root is rounded to be accurate up to the 235 significant digits.

Our analysis has been further confirmed through more test functions that are listed below:

$$f_1(x)=\cos x-x, \alpha=0.739085133215161$$
$$f_2(x)=(\sin^2 x-x^2+1)(\cos 2x+2x^2-3),$$
$$\alpha=1.40449164821534$$

TABLE I
CONVERGENCE FOR

$$f(x)=\left\{x^{10}-\sqrt{3}x^3\cos(\pi x/6)+1/(x^2+1)\right\}(x-1)$$

| $n$ | $x_n$ | $\mid x_n-\alpha\mid$ | $e_{n+1}/{e_n}^2$ | $\eta$ |
|---|---|---|---|---|
| 0 | 0.920000000000000 | 0.800000 | | |
| 1 | 0.991249484161490 | 0.0121355 | 1.367268100 | 3.033 |
| 2 | 0.999794338683744 | 0.000360502 | 2.685871931 | 54251 |
| 3 | 0.999999872081448 | $3.35582\times 1-^{-7}$ | 3.024323988 | |
| 4 | 0.999999999999950 | $2.91278\times 10^{-14}$ | 3.033536758 | |
| 5 | 1.00000000000000 | $2.19446\times 10^{-27}$ | 3.033542510 | |
| 6 | 1.00000000000000 | $1.24557\times 10^{-52}$ | 3.033542510 | |
| 7 | 1.00000000000000 | $4.01279\times 10^{-104}$ | 3.033542510 | |
| 8 | 1.00000000000000 | $4.16489\times 10^{-207}$ | 3.033542510 | |
| 9 | 1.00000000000000 | $-2.89905\times 10^{-400}$ | | |

TABLE II
CONVERGENCE FOR VARIOUS TEST FUNCTIONS.

| $f(x)$ | $m$ | $x_0$ | $e_n$ | $\nu$ | $\eta$ |
|---|---|---|---|---|---|
| $f_1(x)$ | 1 | 0.490 | $6.13024\times 10^{-293}$ | 5 | 0.04875502284 |
| $f_2(x)$ | 2 | 1.290 | $4.51173\times 10^{-253}$ | 8 | 0.7835709502 |
| $f_3(x)$ | 3 | 1.080 | $0.\times 10^{-249}$ | 10 | 5.119146433 |
| $f_4(x)$ | 4 | 2.190 | $1.18904\times 10^{-261}$ | 8 | 0.5369302217 |
| $f_5(x)$ | 5 | 2.270 | $2.41280\times 10^{-398}$ | 9 | 1.11 |
| $f_6(x)$ | 6 | 2.790 | $2.52653\times 10^{-359}$ | 9 | 1.096153846 |
| $f_7(x)$ | 7 | 2.590 | $1.92369\times 10^{-587}$ | 10 | 3.591527519 |
| $f_8(x)$ | 8 | 1.590 | $1.90760\times 10^{-308}$ | 8 | 0.08249684013 |

$f_3(x)=(\sin(\pi x/2\sqrt{2})-x^4+3)(x^2-2)^2$, $\alpha=\sqrt{2}$

$f_4(x)=(x^8-14x^4\sin(\pi x/4)-32)(x^2-4x+4)\log(x-1), \alpha=2.00000000000000$

$f_5(x)=(3x^7-37x^4+208)\sin(\pi x/2)\log[x-1]^3$, $\alpha=2.00000000000000$

$f_6(x)=(e^{(x^2+7x-30)}-1)(x-3)\sin^4\pi x/3$, $\alpha=3.00000000000001$

$f_7(x)=(e^{-x}\sin x+\log[1+(x-\pi)^2])(x-\pi)\sin^3 x(\log[x-\pi+1])^2, \alpha=\pi$

$f_8(x)=(x^2\sin(\pi x/8)+e^{(x-2)^2}-1-2\sqrt{2})(x-2)^3\sin^4(\pi x/2), \alpha=2.00000000000000$

Table 2 shows convergence behavior for the above test functions with the multiplicity $m$, the initial guess $x_0$, the least iteration number $\nu$ and the asymptotic error constant $\eta$. In the future study, we develop extended optimal iteration methods of higher order.

## ACKNOWLEDGMENT

Young Hee Geum was supported by the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (Project No. 2011-0014638).

## REFERENCES

[1] R. G. Bartle, *The Elements of Real Analysis*, 2nd ed., John Wiley & Sons., New York, 1976.
[2] Ward Cheney and David Kincaid, *Numerical Mathematics and Computing*, Brooks/Cole Publishing Company, Monterey, California 1980
[3] S. D. Conte and Carl de Boor, *Elementary Numerical Analysis*, McGraw-Hill Inc., 1980
[4] Qiang Du, Ming Jin, T. Y. Li and Z. Zeng, *"The Quasi-Laguerre Iteration"*, Mathematics of Computation, Vol. 66, No. 217(1997), pp.345-361.
[5] Y. H. Geum, *"The asymptotic error constant of leap-frogging Newtons method locating a simple real zero"*, Applied Mathematics of Computation, Vol. 66, No. 217(2007), pp.345-361.
[6] Y. K. Kim and Y. H. Geum, *" A cubic-order variant of Newton's method for finding multiple roots of nonlinear equations"*, Computer and Mathematics with Applications, Vol. 62(2011), pp.1634-1640
[7] A. Bathi Kasturiarachi, *"Leap-frogging Newton's Method"*, INT. J. MATH. EDUC. SCI. TECHNOL., Vol. 33, No. 4(2002), pp.521-527.
[8] L. D. Petkovic, M. S. Petkovic and D. Zivkovic, *"Hansen-Patrick's Family Is of Laguerre's Type"* , Novi Sad J. Math., Vol. 33, No. 1(2003), pp.109-115.
[9] Kenneth A. Ross, *Elementary Analysis*, Springer-Verlag New York Inc., 1980.
[10] J. Stoer and R. Bulirsh, *Introduction to Numerical Analysis*, pp.244-313, Springer-Verlag New York Inc., 1980.
[11] J. F. Traub, *Iterative Methods for the Solution of Equations*, Chelsea Publishing Company, 1982.
[12] Stephen Wolfram, *The Mathematica Book, 4th ed.*, Cambridge University Press, 1999.