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On Statistical Limit Points in a Fuzzy Valued Metric Space

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Abstract—We introduce the concepts statistical cluster and statistical limit points of a sequence of fuzzy numbers in a fuzzy valued metric space. Then we obtain some inclusion relations between the sets of limit points, statistical limit points and statistical cluster points for a sequence of fuzzy numbers.

Keywords— Statistical convergence; Statistical limit point; Fuzzy valued metric space.

I. Introduction and Background

OVER the past few years the theory of convergence of a sequence of fuzzy numbers has been studied by many authors [1-3,8,11,19]. The first steps towards constructing such convergence theories go back to Matloka's [13] and Kaleva's [12] works. To this end, they used the supremum metric that gives a real (crisp) value for the distance between two fuzzy numbers. On the other hand, via positive fuzzy numbers, it is also possible to define a fuzzy (non-crisp) distance between two fuzzy numbers (as is exemplified by Guangquan [9]), because it is more natural that the distance between two fuzzy numbers is a fuzzy number rather than this distance is a real number.

In this paper, we introduce the concept of statistical convergence of a sequence of fuzzy numbers in a fuzzy metric space which defined by Guangquan [9,10]. Then we compare this definition with the definition of statistical convergence with respect to the supremum metric. We note that this convergence should not be perceived as a generalization of ordinary statistical convergence (see Example 1). Moreover we define the concepts statistical cluster and statistical limit points of a sequence of fuzzy numbers in this fuzzy valued metric space. Finally we obtain some inclusion relations between the sets of limit points, statistical limit points and statistical cluster points for a sequence of fuzzy numbers.

Now we recall some definitions and notations that will be used frequently (see [4-7,10,12,13-18] for more details).

Given an interval A, we denote its endpoints by $\underline{A}$ and $\overline{A}$. We denote by D the set of all closed intervals on the real line $\mathbb{R}$. That is,

$$D := \{A \subset \mathbb{R} : \quad A = [\underline{A}, \overline{A}]\}.$$

For $A, B \in D$ define

$$A \le B \quad \text{iff} \quad \underline{A} \le \underline{B} \quad \text{and} \quad \overline{A} \le \overline{B},$$
$$d(A,B) := \max(|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|).$$

It is easy to see that d defines a metric (Hausdorff metric) on D and (D,d) is a complete metric space. Also "$\le$" is a partial order on D.

A fuzzy number is a function X from $\mathbb{R}$ to $[0,1]$, satisfying

- X is normal, i.e., there exists $x_0 \in \mathbb{R}$ such that $X(x_0) = 1$;
- X is fuzzy convex, i.e., for any $x, y \in \mathbb{R}$ and $\lambda \in [0,1], X(\lambda x + (1-\lambda)y) \ge \min\{X(x), X(y)\}$;
- X is upper semi-continuous;
- the closure of the set $\{x \in \mathbb{R} : X(x) > 0\}$, denoted by X^0, is compact.

These properties imply that for each $\alpha \in (0,1]$, the $\alpha-$level set $X^\alpha := \{x \in \mathbb{R} : X(x) \ge \alpha\} = \left[\underline{X}^\alpha, \overline{X}^\alpha\right]$ is a nonempty compact convex subset of $\mathbb{R}$, as the support X^0 : $X^0 = \lim_{\alpha \to 0^+} X^\alpha$ is We denote the set of all fuzzy numbers by $\mathbf{F}(\mathbb{R})$. Note that the function a_1 defined by

$$a_1(x) := \begin{cases} 1 & ,\text{if } x = a \\ 0 & ,\text{otherwise} \end{cases},$$

where $a \in \mathbb{R}$, is a fuzzy number. By the *decomposition theorem* of fuzzy sets, we have

$$X = \sup_{\alpha \in [0,1]} \alpha \chi_{\left[\underline{X}^\alpha, \overline{X}^\alpha\right]}$$

for every $X \in \mathbf{F}(\mathrm{R})$, where each $\chi_{\left[\underline{X}^{\alpha}, \overline{X}^{\alpha}\right]}$ denotes the characteristic function of the subinterval $\left[\underline{X}^{\alpha}, \overline{X}^{\alpha}\right]$.

Now we recall the *partial order relation* on the set of fuzzy numbers. For $X, Y \in \mathbf{F}(\mathrm{R})$, we write $X \leq Y$, if for every $\alpha \in [0,1]$, the inequality

$$X^{\alpha} \leq Y^{\alpha}$$

holds. We write $X < Y$, if $X \leq Y$, and there exists an $\alpha_0 \in [0,1]$ such that

$$\underline{X}^{\alpha_0} < \underline{Y}^{\alpha_0} \text{ or } \overline{X}^{\alpha_0} < \overline{Y}^{\alpha_0}.$$

If $X \leq Y$ and $Y \leq X$, then $X = Y$. Two fuzzy numbers X and Y are said to be *incomparable* and denoted by $X \not\leq Y$, if neither $X \leq Y$ nor $Y \leq X$ holds. When $X \geq Y$ or $X \not\leq Y$, then we can write $X \not< Y$.

Now let us briefly review the operations of *summation* and *subtraction* on the set of fuzzy numbers. For $X, Y, Z \in \mathbf{F}(\mathrm{R})$, the fuzzy number Z is called the *sum of* X *and* Y, and we write $Z = X + Y$, if $Z^{\alpha} = \left[\underline{Z}^{\alpha}, \overline{Z}^{\alpha}\right] := X^{\alpha} + Y^{\alpha}$ for every $\alpha \in [0,1]$. Similarly, we write $Z = X - Y$, if $Z^{\alpha} = \left[\underline{Z}^{\alpha}, \overline{Z}^{\alpha}\right] := X^{\alpha} - Y^{\alpha}$ for every $\alpha \in [0,1]$.

We define the set of *positive fuzzy numbers* by

$$\mathbf{F}^{+}(\mathrm{R}) := \left\{ X \in \mathbf{F}(\mathrm{R}) : X \geq 0_1 \text{ and } \overline{X}^{1} > 0 \right\}.$$

The map $d_M : \mathbf{F}(\mathrm{R}) \times \mathbf{F}(\mathrm{R}) \to R^{+} \cup \{0\}$ defined as

$$d_M(X,Y) := \sup_{\alpha \in [0,1]} d\left(X^{\alpha}, Y^{\alpha}\right)$$

is called the *supremum metric* on $\mathbf{F}(\mathrm{R})$.

A sequence $X = \{X_n\}$ of fuzzy numbers is said to be *convergent* to the fuzzy number X_0, written as $\lim X_n = X_0$, if for every $\varepsilon > 0$ there exists a positive integer $n_0 = n_0(\varepsilon)$ such that

$$d_M(X_n, X_0) < \varepsilon \quad \text{for every } n > n_0.$$

A fuzzy number λ is called a *limit point* of the sequence $X = \{X_n\}$ of fuzzy numbers provided that there is a subsequence of X that converges to λ. We will denote the set of all limit points of $X = \{X_n\}$ by L_X.

Guangquan [9] introduced the concept of fuzzy distance between two fuzzy numbers as in Definition 1, and thus presented a concrete fuzzy metric in (1.1), which is very similar to an ordinary metric.

Definition 1. *A map* $\rho : \mathbf{F}(\mathrm{R}) \times \mathbf{F}(\mathrm{R}) \to \mathbf{F}(\mathrm{R})$ *is called a fuzzy metric on* $\mathbf{F}(\mathrm{R})$ *provided that the conditions*

(i) $\rho(X,Y) \geq 0_1$,

(ii) $\rho(X,Y) = 0_1$ *if and only if* $X = Y$,

(iii) $\rho(X,Y) = \rho(Y,X)$,

(iv) $\rho(X,Y) \leq \rho(X,Z) + \rho(Z,Y)$ *are satisfied for all* $X, Y, Z \in \mathbf{F}(\mathrm{R})$.

If ρ is a fuzzy metric on the set of fuzzy numbers, then we call the triple $(\mathrm{R}, \mathbf{F}(\mathrm{R}), \rho)$ a *fuzzy metric space*. Guangquan [9] presented an example of a fuzzy metric space via the function d_G defined by

$$d_G(X,Y) := \sup_{\alpha \in [0,1]} \alpha \chi_{\left[\left|\underline{X}^{1} - \underline{Y}^{1}\right|, \sup_{t \in [\alpha,1]} d\left(X^{t}, Y^{t}\right)\right]}. \tag{1.1}$$

Here the map d_G satisfies the conditions (i)-(iv) in Definition 1.

Remark 1. *Let*

$$\mathbf{B}_F := \left\{ K(X,P) : X \in \mathbf{F}(\mathrm{R}), P \in \mathbf{F}^{+}(\mathrm{R}) \right\} \subset \mathbf{P}(\mathbf{F}(\mathrm{R}))$$

where $\mathbf{P}(\mathbf{F}(\mathrm{R}))$ *is the power set of* $\mathbf{F}(\mathrm{R})$ *and* $K(X,P) := \left\{ Z \in \mathbf{F}(\mathrm{R}) : d_G(X,Z) < P, P \in \mathbf{F}^{+}(\mathrm{R}) \right\}$. *Then the set* $\mathbf{B}_F$ *forms a basis of a natural topology on* $\mathbf{F}(\mathrm{R})$, *denoted by* τ_F. *Thus the pair* $(\mathbf{F}(\mathrm{R}), \tau_F)$ *is a topological space.*

Now we investigate the properties of the convergence of a sequence in this topological space. Since this convergence is in the topology τ_F, we will denote it by $\tau_F -$convergence.

Definition 2 ($\tau_F -$convergence)**.** *Let* $X = \{X_n\} \subset \mathbf{F}(\mathrm{R})$ *and* $X_0 \in \mathbf{F}(\mathrm{R})$. *Then* $\{X_n\}$ *is* $\tau_F -$*convergent to* X_0 *and we denote this by*

$$\tau_F - \lim X_n = X_0 \text{ or } \{X_n\} \xrightarrow{\tau_F} X_0 \ (n \to \infty),$$

provided that for any $P \in \mathbf{F}^{+}(\mathrm{R})$ *there exists an* $n_0 = n_0(P) \in \mathrm{N}$ *such that*

$$d_G(X_n, X_0) < P \quad \text{as } n > n_0.$$

Definition 3 ($\tau_F -$limit point)**.** *A fuzzy number* λ *is a* $\tau_F -$*limit point of the sequence* $X = \{X_n\}$ *of fuzzy numbers provided that there is a subsequence of* X *that* $\tau_F -$*converges to* λ. *We denote the set of all* $\tau_F -$*limit points of* $X = \{X_n\}$ *by* $L_X^{\tau_F}$.

Let K be a subset of the set N of positive integers and let us denote the set $\{k \in K : k \leq n\}$ by K_n. Then the *natural* *density* of K is defined by $\delta(K) := \lim_{n\to\infty} \frac{|K_n|}{n}$, where $|K_n|$ denotes the number of elements in K_n. Clearly, a finite subset has natural density zero and we have $\delta(K^c) = 1 - \delta(K)$ whenever $\delta(K)$ exists, where $K^c := \mathrm{N} \setminus K$ is the complement of $K \subset \mathrm{N}$. If $K_1 \subseteq K_2$, then $\delta(K_1) \le \delta(K_2)$. In addition, $\delta(K) \ne 0$ means that $\overline{\delta}(K) := \limsup_{n\to\infty} \frac{|K_n|}{n} > 0.$

A sequence $X = \{X_n\}$ of fuzzy numbers is said to be statistically convergent to the fuzzy number X_0, written as $st - \lim X_n = X_0$, if the set

$$\{n \in \mathrm{N} \,:\, d_M(X_n, X_0) \ge \varepsilon\}$$

has natural density zero for every $\varepsilon > 0$.

Definition 4 (Nonthin subsequence)**.** *If $\{X_{n(j)}\}$ is a subsequence of $X = \{X_n\}$ and $K := \{n(j) \in \mathrm{N} \,:\, j \in \mathrm{N}\}$ then we abbreviate $\{X_{n(j)}\}$ by $\{X\}_K$ which in case $\delta(\{K\}) = 0$ is called a subsequence of density zero or thin subsequence. On the other hand, $\{X\}_K$ is a nonthin subsequence of X if K does not have density zero.*

Definition 5 (Statistical limit point)**.** *The fuzzy number ν is called statistical limit point of the sequence $X = \{X_n\}$ of fuzzy numbers provided that there is a nonthin subsequence of X that converges to ν. Let Λ_X denote the set of statistical limit points of the sequence X.*

Definition 6 (Statistical cluster point)**.** *The fuzzy number μ is called statistical cluster point of the sequence $X = \{X_n\}$ of fuzzy numbers provided that*

$$\overline{\delta}(\{n \in \mathrm{N} \,:\, d_M(X_n, \mu) < \varepsilon\}) > 0$$

for every $\varepsilon > 0$. Let Γ_X denote the set of statistical cluster point of the sequence X.

II. τ_F – STATISTICAL CONVERGENCE

Definition 7 (τ_F – statistical convergence)**.** *Let $X = \{X_n\}$ be a sequence of fuzzy numbers and X_0 be a fuzzy number. The sequence X is said to be τ_F – statistically convergent to X_0, we denote this by*

$$\tau_F \text{ - st - } \lim X_n = X_0,$$

provided that for any $P \in \mathbf{F}^+(\mathrm{R})$, the set

$$\{n \in \mathrm{N} \,:\, d_G(X_n, X_0) \not\prec P\}$$

has natural density zero.

It is clear that if a sequence is τ_F – convergent then it is τ_F – statistically convergent to the same fuzzy number. But the converse of this claim does not hold in general.

Example 1. *It is obvious that the sequence $X = \{X_n\}$ defined by*

$$X_n(x) := \begin{cases} \left.\begin{matrix} 0 & \text{, if } x \in (-\infty, n-1) \cup (n+1, \infty) \\ x-(n-1) & \text{, if } x \in [n-1, n] \\ -x+(n+1) & \text{, otherwise} \end{matrix}\right\} & \text{, if } n = k^2 \ (k \in \mathrm{N}) \\ \left.\begin{matrix} 1 - \frac{nx}{n+1} & \text{, if } x \in [0, 1+\frac{1}{n}] \\ 0 & \text{, otherwise} \end{matrix}\right\} & \text{, otherwise} \end{cases}$$

is τ_F – statistically convergent to the fuzzy number

$$X_0(x) := \begin{cases} 1-x & \text{, if } x \in [0,1] \\ 0 & \text{, otherwise} \end{cases}.$$

On the other hand, since the set $\{n \in \mathrm{N} \,:\, d_G(X_n, X_0) \not\prec P\}$ has infinitely many elements for every $P \in \mathbf{F}^+(\mathrm{R})$, we say that this sequence is not τ_F – convergent.

Theorem 1. *If a sequence $X = \{X_n\}$ of fuzzy numbers τ_F – statistically convergent to the fuzzy number X_0, then this sequence statistically converges to the same fuzzy number X_0 with respect to supremum metric d_M.*

Proof. Assume τ_F -st- $\lim X_n = X_0$. Then the set $\{n \in \mathrm{N} \,:\, d_G(X_n, X_0) \not\prec \varepsilon_1\}$ has natural density zero for every $\varepsilon_1 \in \mathbf{F}^+(\mathrm{R})$. Fix $\varepsilon > 0$. Then we have

$$\delta\left(\left\{n \in \mathrm{N} \,:\, \chi_{\left[\left|\underline{X_n}^1 - \underline{X_0}^1\right|, \sup_{t\in[\alpha,1]} d\left(X_n^t, X_0^t\right)\right]} \not\prec \varepsilon_1\right\}\right) = 0$$

for every $\alpha \in [0,1]$. It is clear that the inclusion

$$\left\{n \in \mathrm{N} \,:\, \chi_{\left[\left|\underline{X_n}^1 - \underline{X_0}^1\right|, \sup_{t\in[\alpha,1]} d\left(X_n^t, X_0^t\right)\right]} \not\prec \varepsilon_1\right\} \supseteq \left\{n \in \mathrm{N} \,:\, \sup_{t\in[\alpha,1]} d\left(X_n^t, X_0^t\right) \ge \varepsilon\right\}$$

holds for every $\alpha \in [0,1]$, i.e., we have

$$\left\{n \in \mathrm{N} \,:\, \chi_{\left[\left|\underline{X_n}^1 - \underline{X_0}^1\right|, \sup_{t\in[\alpha,1]} d\left(X_n^t, X_0^t\right)\right]} \not\prec \varepsilon_1\right\} \supseteq \{n \in \mathrm{N} \,:\, d_M(X_n, X_0) \ge \varepsilon\}. \qquad (2.1)$$

The inclusion (2.1) say that

$$\delta(\{n \in \mathrm{N} \,:\, d_M(X_n, X_0) \ge \varepsilon\}) = 0$$

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because left hand side of the inclusion (2.1) has natural density zero.

The converse of this theorem does not valid in general as can be seen by the following example.

Example 2. *Define the sequence* $X=\{X_n\}$ *by*

$$X_n(x):=\begin{cases}\left.\begin{cases}x-\left(2+\frac{1}{n}\right) & ,\text{if } x\in\left(2+\frac{1}{n},3+\frac{1}{n}\right)\\ \left(4+\frac{1}{n}\right)-x & ,\text{if } x\in[3+\frac{1}{n},4+\frac{1}{n})\\ 0 & ,\text{otherwise}\end{cases}\right\} & ,\text{if } n\neq k^2\ (k\in\mathbb{N})\\ \left.\begin{cases}\frac{n(9-x)+1}{n+1} & ,\text{if } x\in[8,9+\frac{1}{n}]\\ 0 & ,\text{otherwise}\end{cases}\right\} & ,\text{otherwise}\end{cases}$$

and define the fuzzy number X_0 *by*

$$X_0(x):=\begin{cases}x-2 & ,\text{if } x\in(2,3)\\ 4-x & ,\text{if } x\in[3,4).\\ 0 & ,\text{otherwise}\end{cases}$$

Then $st\text{-}\lim X_n=X_0$. *But* τ_F *-st-* $\lim X_n$ *does not exist. Now we show this claim. Define*

$$P(x):=\begin{cases}0 & ,x\in(-\infty,0]\cup[2,\infty)\\ x & ,x\in(0,1]\\ 2-x & ,\text{otherwise}\end{cases}.$$

Then we have

$$d_G(X_n,X_0)=\sup_{\alpha\in[0,1]}\alpha\chi_{\left[\left|\underline{X}_n^{\,1}-\underline{X}_0^{1}\right|,\ \sup_{t\in[\alpha,1]}d\left(X_n^t,X_0^t\right)\right]}$$
$$=\sup_{\alpha\in[0,1]}\alpha\chi_{\left[\frac{1}{n},\frac{1}{n}\right]}$$

if $n\neq k^2$ *(* $k\in\mathbb{N}$ *). Otherwise we have* $d_G(X_n,X_0)=\sup_{\alpha\in[0,1]}\alpha\chi_{\left[0,\frac{1}{n}\right]}$. *Hence we get* $P\nprec d_G(X_n,X_0)$ *if* $n\neq k^2$, *otherwise* $d_G(X_n,X_0)<P$. *Consequently we have* τ_F *-st-* $\lim X_n\neq X_0$.

III. τ_F -STATISTICAL LIMIT POINTS

Definition 8 (τ_F -statistical limit point)**.** *Let* $X=\{X_n\}$ *be a sequence of fuzzy numbers and* ν *be a fuzzy number. The number* ν *is called* τ_F *-statistical limit point of the sequence* X *if there is a nonthin subsequence of* X *that* τ_F *-converges to* ν. *We denote the set of all* τ_F *-statistical limit points of the sequence* X *by* $\Lambda_X^{\tau_F}$.

Definition 9 (τ_F -statistical cluster point)**.** *Let* $X=\{X_n\}$ *be a sequence of fuzzy numbers and* μ *be a fuzzy number. The number* μ *is called* τ_F *-statistical cluster point of the sequence* X *if the set*

$$\{n\in\mathbb{N}\ :\ d_G(X_n,\mu)<P\}$$

has no natural density zero for every $P\in\mathbf{F}^+(\mathbb{R})$. *Let* $\Gamma_X^{\tau_F}$ *denote the set of all* τ_F *-statistical cluster points of the sequence* X.

Now we give an illustrative example.

Example 3. *Define the sequence* $X=\{X_n\}$ *by*

$$X_n(x):=\begin{cases}\left.\begin{cases}1-\frac{nx}{n+1} & ,\text{if } x\in[0,1+\frac{1}{n}]\\ 0 & ,\text{otherwise}\end{cases}\right\} & ,\text{if } n \text{ is an even number and } n\neq k^2\ (k\in\mathbb{N})\\ \left.\begin{cases}\frac{n(9-x)+1}{n+1} & ,\text{if } x\in[8,9+\frac{1}{n}]\\ 0 & ,\text{otherwise}\end{cases}\right\} & ,\text{if } n \text{ is an even number and } n=k^2\ (k\in\mathbb{N})\\ \left.\begin{cases}x-\left(2+\frac{1}{n}\right) & ,\text{if } x\in\left(2+\frac{1}{n},3+\frac{1}{n}\right)\\ \left(4+\frac{1}{n}\right)-x & ,\text{if } x\in[3+\frac{1}{n},4+\frac{1}{n})\\ 0 & ,\text{otherwise}\end{cases}\right\} & ,\text{if } n \text{ is an odd number and } n\neq k^2\ (k\in\mathbb{N})\\ \left.\begin{cases}x-\left(5+\frac{1}{n}\right) & ,\text{if } x\in[5+\frac{1}{n},6+\frac{1}{n}]\\ 0 & ,\text{otherwise}\end{cases}\right\} & ,\text{if } n \text{ is an odd number and } n=k^2\ (k\in\mathbb{N})\end{cases}.$$

Define

$$\mu_0(x):=\begin{cases}1-x & ,\text{if } x\in[0,1]\\ 0 & ,\text{otherwise}\end{cases},$$

$$\nu_0(x):=\begin{cases}x-2 & ,\text{if } x\in(2,3)\\ 4-x & ,\text{if } x\in[3,4),\\ 0 & ,\text{otherwise}\end{cases}$$

$$\gamma_0(x):=\begin{cases}x-5 & ,\text{if } x\in[5,6]\\ 0 & ,\text{otherwise}\end{cases}$$

and

$$\xi_0(x):=\begin{cases}9-x & ,\text{if } x\in[8,9]\\ 0 & ,\text{otherwise}\end{cases}.$$

Hence we obtain

$$L_X=\{\mu_0,\nu_0,\gamma_0,\xi_0\},$$
$$\Lambda_X=\Gamma_X=\{\mu_0,\nu_0\},$$
$$L_X^{\tau_F}=\{\mu_0,\xi_0\},$$
$$\Lambda_X^{\tau_F}=\Gamma_X^{\tau_F}=\{\mu_0\}.$$

Since τ_F -convergence implies the convergence with respect to supremum metric d_M, it is clear that $L_X^{\tau_F}\subset L_X$ and $\Lambda_X^{\tau_F}\subset\Lambda_X$. Now we prove the relations between the sets of statistical cluster points:

Theorem 2. *We have* $\Gamma_X^{\tau_F}\subset\Gamma_X$ *for a sequence* $X=\{X_n\}$ *of fuzzy numbers.*

Proof. Take $\mu\in\Gamma_X^{\tau_F}$. By definition, we get

$$\delta(\{n\in\mathbb{N}\ :\ d_G(X_n,\mu)<P\})\neq 0$$

for every $P\in\mathbf{F}^+(\mathbb{R})$. Fix $\varepsilon>0$. Then we can write

$$\delta(\{n \in \mathrm{N} : d_G(X_n, \mu) < \varepsilon_1\}) \neq 0,$$

since $\varepsilon_1 \in \mathbf{F}^+(\mathrm{R})$. From definition of the metric d_G, we have

$$\delta\left(\left\{n \in \mathrm{N} : \sup_{t\in[\alpha,1]} d\left(X_n^t, \mu^t\right) < \underline{\varepsilon_1}^t = \varepsilon\right\}\right) \neq 0$$

for every $\alpha \in [0,1]$, i.e., we get

$$\delta(\{n \in \mathrm{N} : d_M(X_n, \mu) < \varepsilon\}) \neq 0$$

by the definition of supremum metric d_M. Since the number ε is arbitrary, the proof of theorem is completed.

Theorem 3. *We have $\Gamma_X^{\tau_F} \subset L_X^{\tau_F}$ for a sequence $X = \{X_n\}$ of fuzzy numbers.*

Proof. Take $\mu \in \Gamma_X^{\tau_F}$. Fix $P \in \mathbf{F}^+(\mathrm{R})$. Then we have $\delta(\{n(j) \in \mathrm{N} : d_G(X_{n(j)}, \mu) < P\}) \neq 0$. Define $\{X\}_K$ by a nonthin subsequence of X such that

$$K = K(P) := \{n \in \mathrm{N} : d_G(X_n, \mu) < P\}$$

and $\delta(K) \neq 0$. Then there exists a subset $L \subset K$ such that $\tau_F\text{-}\lim_{n\in L,\, n\to\infty} X_n = \mu$, where the set L has infinitely many elements. Therefore we get $\mu \in L_X^{\tau_F}$.

The converse of this theorem does not hold in general as can be seen in Example 3.

Theorem 4. *We have $\Lambda_X^{\tau_F} \subset \Gamma_X^{\tau_F}$ for a sequence $X = \{X_n\}$ of fuzzy numbers.*

Proof. Assume $\nu \in \Lambda_X^{\tau_F}$. Then there exists a set $K := \{n(j) \in \mathrm{N} : j \in \mathrm{N}\}$ such that $\overline{\delta}(K) = l > 0$ and

$\tau_F\text{-}\lim_{j\to\infty} X_{n(j)} = \mu.$ Fix $P \in \mathbf{F}^+(\mathrm{R})$. Hence the inclusion

$$\begin{aligned}\{n \in \mathrm{N} : d_G(X_n, \mu) < P\} &\supseteq \{n(j) \in \mathrm{N} : d_G(X_{n(j)}, \mu) < P\} \\ &= K \setminus \{\{n(j) \in \mathrm{N} : d_G(X_{n(j)}, \mu) \geq P\} \cup \{n(j) \in \mathrm{N} : d_G(X_{n(j)}, \mu) \nless P\}\}\end{aligned}$$

holds. Here the sets $\{n(j) \in \mathrm{N} : d_G(X_{n(j)}, \mu) \geq P\}$ and $\{n(j) \in \mathrm{N} : d_G(X_{n(j)}, \mu) \nless P\}$ has finite many elements. Hence we have

$$\begin{aligned}\overline{\delta}(\{n \in \mathrm{N} : d_G(X_n, \mu) < P\}) &\geq \overline{\delta}(K) \\ &\quad - \overline{\delta}(\{n(j) \in \mathrm{N} : d_G(X_{n(j)}, \mu) \geq P\} \cup \{n(j) \in \mathrm{N} : d_G(X_{n(j)}, \mu) \nless P\}) \\ &= l.\end{aligned}$$

Therefore we get $\delta(\{n \in \mathrm{N} : d_G(X_n, \mu) < P\}) \neq 0$.

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