Vibration control of multi walled nanosensor by piezoelectric and electrostatic actuator using nonlocal and surface/interface parameters

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Abstract: - **In this work, vibration control of multi walled piezoelectric nanosensor (MWPENS) using nonclassical theories of nonlocal (NLT), nonlocal strain gradient (NSGT) and Gurtin–Murdoch surface/interface (GMSIT) approaches are presented. The nanosensor is embedded in direct nonlinear electrostatic voltage DC, harmonic excitation, structural damping, two piezoelectric layers and nonlinear van der Waals (vdW) force. Hamilton's principle and Galerkin technique respectively are used to obtain the governing equations and boundary conditions and to solve the equation of motion. For this work, effects of surface/interface energy, size and, material length scale parameters on pull-in voltage $\bar{V}_{DC}$ and dimensionless natural frequency (DNF) are consided and nonclassical theories compared with classical theory. It is concluded that ignoring nonclassical effects lead to inaccurate results in vibrational response of the MWPENS. In all boundary condition, S/I effects lead to increasing of MWPENS stiffness leads to more DC voltage to reach the pull-in instability and other nonclassical effects lead to decreasing of MWPENS stiffness and as a result decreasing of DNF. Also, with ignoring the surface/interface density $\rho^{I,S}$ and Lame's constants $\mu^{I,S}$, respectively the system will have a maximum and minimum DNF than the other parameters and MWPENS respectively will later and sooner than other parameters reach the pull-in voltage.**

Key-words: - **Vibration control; Multi walled piezoelectric nanosensor; Surface/interface effect; Nonlocal effect; Nonlocal strain gradient theory; Pull-in voltage; Electrostatic excitation.**

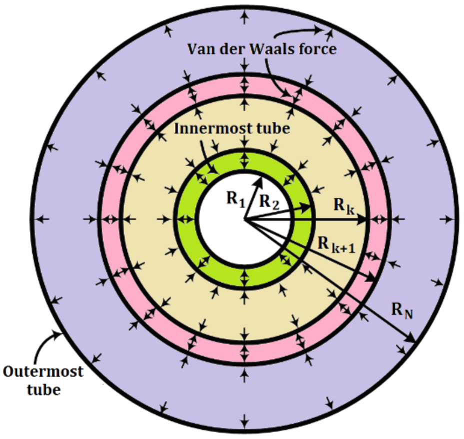
I. INTRODUCTION

Nanotechnology is a branch of science which encompasses numerous fields of technology and science such as nano structures, especially piezoelectric nano sensors/resonators which are widely used in modern engineering and have received considerable attention from researchers around the world, due to their unique features and widespread applications [1, 2]. On the other hand, due to excessive use of nanosensor, especially piezoelectric nanosensor in vibration devices, modeling their mathematical modeling and vibration behaviors are essential. In spite of a large number of studies on nano sensors dynamics, just a very small number of these works is related to the vibration and stability analysis of piezoelectric nanosensor based on multi walled nano shell (MWNS) with nonclassical approaches. For these reasons, nonclassical theories such as nonlocal [3], strain gradient [4] and Gurtin–Murdoch surface/interface [5] theories are presented to investigate vibration analysis of nano-structures such as piezoelectric sensors that has become one of the attractive research areas in nanomechanics. In analysis of nanostructures, Zamani *et al.* utilized piezoelectric layer and nanotechnology to investigate seismic response of smart nanocomposite cylindrical shell conveying fluid flow [6]. Non-linear response of multiwall carbon nanotube reinforced laminated composite beam is studied by Lal and Markad using Halpin–Tsai model, finite element method and minimum potential energy approach [7]. Balubaid *et al.* investigate free vibrational behavior of FG nano-plate using the nonlocal two variables integral refined plate theory [8]. Also Asghar *et al.* utilized Eringen's nonlocal effects to predicte frequency spectrum of double-walled carbon nanotubes [9]. Avramov used Sanders–Koiter and nonlocal approaches to investigate nonlinear vibrations of single-walled carbon nanotubes [10]. Ke *et al.* employed nonlocal piezoelectricity model for nonlinear vibration analysis of the piezoelectric nanobeam [11]. Farajpour *et al.* shown that by considering the length scale effect, the frequency ratio of magneto electro elastic nanoplate decreases [12]. In the work done by Arefi, it has been shown that with increase of nonlocal parameter, due to decreasing of the nano shell stiffness, the rotations, displacements and transverse deflection are increased to change of nonlocal effect [13]. Based on SGT, buckling and postbuckling of nanobeams were studied by Li and Hu based on the nonlocal stain gradient theory and size dependency was presented significant [14]. Ebrahimi and Barati studied vibration characteristics of hydro-thermally affected functionally graded viscoelastic nano-beams subjected to viscoelastic foundation by nonlocal strain gradient theory [15]. According to Gurtin-Murdoch approach of surface/interface elasticity and nonlocal strain gradiant theories, recently, Hashemi Kachapi *et al.* presented surface/interface effects to investigate linear and nonlinear vibration analysis of multi-walled piezoelectric nanostructures [16-25]. Fang *et al.* used surface energy effect to investigate nonlinear vibration behavior of piezoelectric nano structures [26, 27]. Arani *et al.* investigated nonlinear vibration of double-walled nano sheet with small scale and surface effects using nonlocal and surface piezoelasticity theories [28].

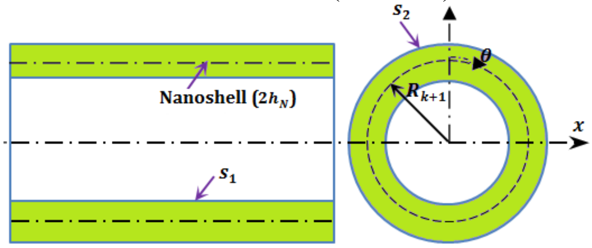
To the best knowledge of the author that vibration control of electrostatically actuated multi walled piezoelectric nanosensor considering simulants effects of nonclassical theories, i.e. nonlocal, nonlocal strain gradient and Gurtin–Murdoch surface/interface theories has not been studied yet. In the present study, the effects of surface/interface energy, size and material length scale parameters on pull-in voltage $\bar{V}_{DC}$ and dimensionless natural frequency in multi walled piezoelectric nanosensor subjected to nonlinear van der Waals force, electrostatic and harmonic excitations and structural damping. For this purpose, nonclassical theories compared with classical theory CT.

II. MATHEMATICAL FORMULATION

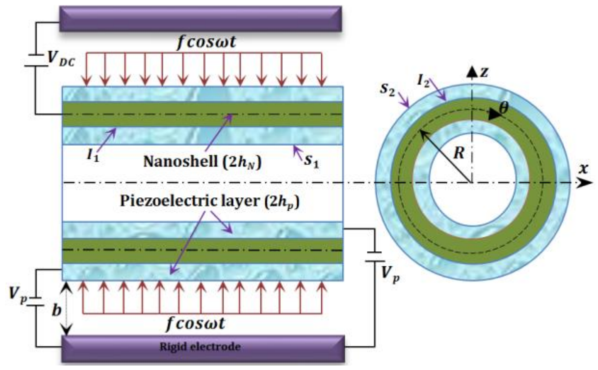
A schematic diagram of multi walled piezoelectric nanosensor embedded with two piezoelectric layers in outer walled is shown in Figure 1 (a-c). This nanostructure subjected to the combined electrostatic force with direct electric voltage (V_{DC}) and harmonic excitation with amplitude f and angular frequency ω and also van der Waals force with linear and nonlinear terms. All of the physical and geometrical properties of the mentioned nanostructures can be seen in reference Hashemi Kachapi *et al.* [17, 25].



(a) Illustration of vdW forces between two adjacent tubes of a multiple shell cross section of a multi walled carbon nanotube (MWCNT)



(b) Modeling of $1 \dots k+1$ tube of MWCNT with surface model



(c) Modeling of last tube of MWCNT as a piezoelectric nanosensor with surface/interface model

Fig. 1 Multi walled piezoelectric nanosensor

A. Nonlocal strain gradient and surface/interface theories

The nonlocal strain gradient stress and electric displacement for the surface/interface of the piezoelectric materials may be represented as

$$(1-\mu\nabla^2)\sigma_{ij}^{NSG(s/I)} = (1-\eta\nabla^2)\left(C_{ijkl}\varepsilon_{kl} - e_{ijk}\bar{E}_k\right), \tag{1}$$

$$(1-\mu\nabla^2)D_k^{NSG(s/I)} = (1-\eta\nabla^2)\left(e_{kli}\varepsilon_{kl} - \eta_{ij}\bar{E}_j\right), \tag{2}$$

The terms $\sigma_{ij}^{NSG(s/I)}$ and $D_k^{NSG(s/I)}$ are the nonlocal surface/interface stress tensor and electric displacement, respectively. All of the piezoelectric parameters (the materials and geometrical parameters) are neglected for first layer and to be zero and all of the materials and geometrical parameters of nanoshell in the first layer are similar to the second layer of nanostructure. The normal stresses σ_{xx} and $\sigma_{\theta\theta}$, Eqs. (1) and (2) can be rewrite ten as

$$\sigma_{xx(N,p)} = C_{11(N,p)}\varepsilon_{xx} + C_{12(N,p)}\varepsilon_{\theta\theta} - e_{31p}\bar{E}_{xp} + \frac{\upsilon_{(N,p)}\sigma_{zz(N,p)}}{1-\upsilon_{(N,p)}}, \tag{3}$$

$$\sigma_{\theta\theta(N,p)} = C_{21(N,p)}\varepsilon_{xx} + C_{22(N,p)}\varepsilon_{\theta\theta} - e_{32p}\bar{E}_{\theta p} + \frac{\upsilon_{(N,p)}\sigma_{zz(N,p)}}{1-\upsilon_{(N,p)}}, \tag{4}$$

$$\sigma_{x\theta(N,p)} = C_{66(N,p)}\gamma_{x\theta}, \tag{5}$$

where σ_{zz} can be rewritten as

$$\sigma_{zz(N,p)} = \frac{z}{h_{Nn}+h_{p2}}\left((\tau_0^{ps}+\tau_0^{NI})\left(\frac{\partial^2 w}{\partial x^2} + \frac{1}{R^2}\frac{\partial^2 w}{\partial\theta^2}\right) - (\rho^{ps}+\rho^{NI})\frac{\partial^2 w}{\partial t^2} \right), \tag{6}$$

Due to existence interaction between the elastic surface and bulk material, in-plane forces in different directions act on the MWPENS. The resulting in-plane loads for piezoelectric nanostructures lead to surface/interface stresses and electric displacement which can be defined using surface/interface constitutive equations as follows

$$(1-\mu\nabla^2)\sigma_{xx(N,p)}^{NSG\left(\frac{s}{I}\right)} = (1-\eta\nabla^2)\times\left(C_{11(N,p)}\varepsilon_{xx} + C_{12(N,p)}\varepsilon_{\theta\theta} - e_{31p}\bar{E}_{xp} + \frac{\upsilon_{(N,p)}\sigma_{zz}}{1-\upsilon_{(N,p)}} \right), \tag{7}$$

$$(1-\mu\nabla^2)\sigma_{\theta\theta(N,p)}^{NSG\left(\frac{s}{I}\right)} = (1-\eta\nabla^2)\times\left(C_{21(N,p)}\varepsilon_{xx} + C_{22(N,p)}\varepsilon_{\theta\theta} - e_{32p}\bar{E}_{\theta p} + \frac{\upsilon_{(N,p)}\sigma_{zz}}{1-\upsilon_{(N,p)}} \right), \tag{8}$$

$$(1-\mu^2\nabla^2)\sigma_{x\theta(N,p)}^{NSG\left(\frac{s}{I}\right)} = (1-l^2\nabla^2)\times\left(C_{66(N,p)}\gamma_{x\theta}\right), \tag{9}$$

Also all coefficients and phrases of Eqs. (1)- (9) such as displacement fields, nonlinear deflection and curvatures, relations of Gurtin–Murdoch surface/interface elasticity theory and etc. can be seen in full detail in reference (Hashemi Kachapi *et al.* [17, 25], Amabili [29]).

B. Governing equations

In this section, the governing equations of motion and corresponding boundary conditions of the piezoelectric shell are obtained by applying the following Hamilton principle:

$$\int_0^t \left(\delta T_n - \delta\pi_n + \delta w_{vdw} + \delta w_{vf} + \delta w_e + \delta W_f\right) dt = 0, \tag{10}$$

Where δT_n, $\delta\pi_n$, δw_{vdw}, δw_{vf}, δw_e and δw_f respectively are the first variation of strain energy, kinetic energy, nonlinear van der Waals interaction, viscoelastic foundation, and nonlinear electrostatic force and harmonic excitation. The first variation of strain energy can be obtained as

$$\delta\pi_n = \int_0^L\int_0^{2\pi}\left\{\begin{array}{c} \int_{-h_{Nn}}^{h_{Nn}}\left(\sigma_{ijNn}\delta\varepsilon_{ijn}\right)dz + \\ \int_{-h_{N2}-h_{p2}}^{-h_{N2}}\begin{pmatrix}\sigma_{ijp2}\delta\varepsilon_{ij2}\\ -\bar{E}_{zp2}\delta D_{zp2}\end{pmatrix} \\ +\int_{h_{N2}}^{h_{N2}+h_{p2}}\begin{pmatrix}\sigma_{ijp2}\delta\varepsilon_{ij2}\\ -\bar{E}_{zp2}\delta D_{zp2}\end{pmatrix}dz \\ +\left(\sigma_{ij2}^{S}\delta\varepsilon_{ij2} - \bar{E}_{zp2}\delta D_{i2}^{S}\right)\times \\ \left(R_2 + h_{N2} + h_{p2}\right) \\ +\left(\sigma_{ij2}^{S}\delta\varepsilon_{ij2} - \bar{E}_{zp2}\delta D_{i2}^{S}\right) \\ +\left(\sigma_{ij2}^{I}\delta\varepsilon_{ij2}\right)\left(R_2 - h_{N2}\right) \\ +\left(\sigma_{ijn}^{I}\delta\varepsilon_{ijn}\right)\left(R_n + h_{Nn}\right)\end{array}\right\}$$

$$\times R_n d\theta dx = \int_0^L\int_0^{2\pi}\left\{\begin{array}{c} N_{xxn}\left(\frac{\partial\delta u_n}{\partial x} + \frac{\partial w_n}{\partial x}\frac{\partial\delta w_n}{\partial x}\right) \\ +N_{\theta\theta n}\begin{pmatrix}\frac{1}{R_n}\left(\frac{\partial\delta v_n}{\partial\theta} + \delta w_n\right)\\ +\frac{1}{R_n^2}\left(\frac{\partial w_n}{\partial\theta}\frac{\partial\delta w_n}{\partial\theta}\right)\end{pmatrix} \\ +N_{x\theta n}\begin{pmatrix}\frac{1}{R_n}\frac{\partial\delta u_n}{\partial\theta} + \frac{\partial\delta v_n}{\partial x}\\ +\frac{1}{R_n}\frac{\partial\delta w_n}{\partial x}\frac{\partial w_n}{\partial\theta} + \\ \frac{1}{R_n}\frac{\partial w_n}{\partial x}\frac{\partial\delta w_n}{\partial\theta}\end{pmatrix} \\ -M_{xxn}\left(\frac{\partial^2\delta w_n}{\partial x^2}\right) - \\ M_{\theta\theta n}\left(\frac{1}{R_n^2}\frac{\partial^2\delta w_n}{\partial\theta^2}\right) - \\ M_{x\theta n}\left(\frac{2}{R_n}\frac{\partial^2\delta w_n}{\partial x\partial\theta}\right)\end{array}\right\} \times R_n d\theta dx \tag{11}$$

In Eq. (11), the forces (N) and moment (M) resultants are defined in reference Hashemi Kachapi *et al.* [17, 25]. Also, the first variation of kinetic energy can be written as

$$\delta T_n = -\iint I_n\begin{pmatrix}\left(\frac{\partial^2 u_n}{\partial t^2}\right)\delta u_n + \\ \left(\frac{\partial^2 v_n}{\partial t^2}\right)\delta v_n \\ +\left(\frac{\partial^2 w_n}{\partial t^2}\right)\delta w_n\end{pmatrix}R_n d\theta dx \tag{12}$$

where

$$I_n = \int_{-h_{Nn}}^{h_{Nn}}\rho_{Nn}\,dz + \int_{-h_{Nn}-h_{p2}}^{-h_{Nn}}\rho_{p2}\,dz + \int_{h_{Nn}}^{h_{Nn}+h_{p2}}\rho_{p2}\,dz + \rho_n^{S,I} = 2\rho_{Nn}h_{Nn} + 2\rho_{p2}h_{p2} + 2\rho_n^{ps} + 2\rho_n^{NI} \tag{13}$$

The first variation of the work done by the nonlinear van der Waals interaction, viscoelastic foundation, nonlinear electrostatic force and the external harmonic excitation, for example with three walled piezoelectric nanosensor (TWPENS), respectively, can be expressed as (He *et al.* [30], Jafari *et al.* [31], Farokhi [32], Hashemi Kachapi *et al.* [25])

$$\delta W_{vdw} = \int_0^L\int_0^{2\pi}\int_0^{w_1}\begin{pmatrix}C_{vdw(12)}^{L}(w_2 - w_1)\\ +C_{vdw(12)}^{NL}(w_2 - w_1)^3\end{pmatrix} \times \delta w_1 R_1 d\theta dx$$
$$\int_0^L\int_0^{2\pi}\int_0^{w_2}\begin{pmatrix}C_{vdw(23)}^{L}\times\\ (w_3 - w_2)\\ +C_{vdw(23)}^{NL}\times\\ (w_3 - w_2)^3\\ -\left(\frac{R_1}{R_2}\right)\times\\ \begin{pmatrix}C_{vdw(12)}^{L}\times\\ (w_2 - w_1) + \\ C_{vdw(12)}^{NL}\times\\ \begin{pmatrix}w_2 - \\ w_1\end{pmatrix}^3\end{pmatrix}\end{pmatrix} \times \delta w_2 R_2 d\theta dx - \left(\frac{R_2}{R_3}\right)\times \tag{14}$$
$$\int_0^L\int_0^{2\pi}\int_0^{w_3}\begin{pmatrix}C_{vdw(32)}^{L}\times\\ (w_3 - w_2)\\ +C_{vdw(32)}^{NL}\times\\ (w_3 - w_2)^3\end{pmatrix}\delta w_3 \; R_3 d\theta dx,$$

$$\delta W_{vf} = -\int_0^L \int_0^{2\pi} \int_0^{w_3} C_w \frac{\partial w_3}{\partial t} \delta w_3 R_3 d\theta dx, \tag{15}$$

$$\delta W_e = \int_0^L \int_0^{2\pi} \int_0^{w_3} \frac{\pi \Upsilon V_{DC}^2}{\sqrt{(b_2 - w_3)(2R_3 + b_3 - w_3)}} \times \left[\cosh^{-1}\left(1 + \frac{b_3 - w_3}{R_3}\right)\right]^2 \times \delta w_3 R_3 d\theta dx, \tag{16}$$

$$\delta W_f = \int_0^L \int_0^{2\pi} \int_0^{w_3} (f cos\omega t)\, \delta w_3 R_3 d\theta dx, \tag{17}$$

C_{vdwn}^{L} and C_{vdwn}^{NL} are the linear and nonlinear vdW interaction coefficients, respectively $C_w = 2\zeta_w M \omega_n$ is structural damping coefficient and ζ_w, M.and ω_n respectively are damping factor, total mass of nanostructure and natural frequency. The damping factor ζ_w, is the actual damping ratio of a system to the critical damping constant. The Laplace operator is $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{R^2}\frac{\partial^2}{\partial \theta^2}$. Finally, for electrostatic force, electrode distance to nanoshell and the air permittivity are b_3 and $\Upsilon = 8.85 \times 10^{-12} C^2 N^{-1} m^{-2}$, respectively and also f and ω are the amplitude and angular frequency of the external excitation.

By substituting Eqs. (11-17) into Eq. (10), integrating the displacement gradients by parts and setting the coefficients $\delta u, \delta v$ and δw to zero, the governing equations of motion and boundary conditions for MWPENS respectively are obtained as follows;

Equations of motion:

$$\delta u_n: \quad \frac{\partial N_{xn}}{\partial x} + \frac{1}{R_n}\frac{\partial N_{x\theta n}}{\partial \theta} = I_n \frac{\partial^2 u_n}{\partial t^2}, \tag{18}$$

$$\delta v_n: \quad \frac{\partial N_{x\theta n}}{\partial x} + \frac{1}{R_n}\frac{\partial N_{\theta n}}{\partial \theta} = I_n \frac{\partial^2 v_n}{\partial t^2}, \tag{19}$$

$$\begin{aligned}\delta w_n: \quad & \frac{\partial^2 M_{xn}}{\partial x^2} + \frac{2}{R_n}\frac{\partial^2 M_{x\theta n}}{\partial x \partial \theta} + \frac{1}{R_n^2}\frac{\partial^2 M_{\theta n}}{\partial \theta^2} \\ & - \frac{N_{\theta n}}{R_n} + N_{xn}\frac{\partial^2 w_n}{\partial x^2} + \frac{\partial N_{xn}}{\partial x}\frac{\partial w_n}{\partial x} \\ & + \frac{N_{\theta n}}{R_n^2}\frac{\partial^2 w_n}{\partial \theta^2} \\ & + \frac{1}{R_n^2}\frac{\partial N_{\theta n}}{\partial \theta}\frac{\partial w_n}{\partial \theta} + \frac{2}{R_n} N_{x\theta n}\frac{\partial^2 w_n}{\partial x \partial \theta} \\ & + \frac{1}{R_n}\frac{\partial N_{x\theta n}}{\partial x}\frac{\partial w_n}{\partial \theta} + \frac{1}{R_n}\frac{\partial N_{x\theta n}}{\partial \theta}\frac{\partial w_n}{\partial x} \\ & = I_n \frac{\partial^2 w_n}{\partial t^2} + S_n \\ & - \frac{\pi \Upsilon V_{DC}^2}{\left(\sqrt{(b_2 - w_2)(2R_2 + b_2 - w_2)} \times \left[\cosh^{-1}\left(1 + \frac{b_2 - w_2}{R_2}\right)\right]^2\right)} \\ & - f cos\omega t,,\end{aligned} \tag{20}$$

where S_n for the inner and outer layer, respectively, are:

$$S_1 = -\left(\begin{matrix} C_{vdw(12)}^{L}(w_2 - w_1) + \\ C_{vdw(12)}^{NL}(w_2 - w_1)^3 \end{matrix}\right),) \tag{21}$$

$$S_2 = \left(\begin{matrix} -C_{vdw(23)}^{L}(w_3 - w_2) \\ -C_{vdw(23)}^{NL}(w_3 - w_2)^3 \\ +\left(\frac{R_1}{R_2}\right)\left(\begin{matrix} C_{vdw(12)}^{L}(w_2 - w_1) \\ +C_{vdw(12)}^{NL}(w_2 - w_1)^3 \end{matrix}\right) \end{matrix}\right) \tag{22}$$

$$S_3 = \left(\begin{matrix} \left(\frac{R_2}{R_3}\right)\left(\begin{matrix} C_{vdw(32)}^{L}(w_3 - w_2) + \\ C_{vdw(32)}^{NL}(w_3 - w_2)^3 \end{matrix}\right) + \\ C_w \frac{\partial w_3}{\partial t} \end{matrix}\right) \tag{23}$$

and the associated boundary conditions are

$$\delta u_n = 0 \quad or \quad N_{xn} n_x + \frac{1}{R_n} N_{x\theta n} n_\theta = 0, \tag{24}$$

$$\delta v_n = 0 \quad or \quad N_{x\theta n} n_x + \frac{1}{R_n} N_{\theta n} n_\theta = 0, \tag{25}$$

$$\begin{aligned}& \delta w_n = 0 \quad or \\ & \left(\begin{matrix} \frac{\partial M_{xn}}{\partial x} + \frac{1}{R_n}\frac{\partial M_{x\theta n}}{\partial \theta} + \\ N_{xx}\frac{\partial w_n}{\partial x} + \frac{N_{x\theta}}{R_n}\frac{\partial w_n}{\partial \theta} \end{matrix}\right) n_x \\ & + \left(\begin{matrix} \frac{1}{R_n}\frac{\partial M_{x\theta n}}{\partial x} + \frac{1}{R_n^2}\frac{\partial M_{\theta n}}{\partial \theta} + \\ \frac{N_{x\theta n}}{R_n}\frac{\partial w_n}{\partial x} + \frac{N_{\theta n}}{R_n^2}\frac{\partial w_n}{\partial \theta} \end{matrix}\right) n_\theta = 0,\end{aligned} \tag{26}$$

$$\frac{\partial w_n}{\partial x} = 0 \quad or \quad M_{xn} n_x + \frac{1}{R_n} M_{x\theta n} n_\theta = 0, \tag{27}$$

$$\begin{aligned}& \frac{\partial w_n}{\partial \theta} = 0 \quad or \\ & \frac{1}{R_n} M_{x\theta n} n_x + \frac{1}{R_n^2} M_{\theta n} n_\theta = 0,\end{aligned} \tag{28}$$

Considering nonlocal and material length scale effects and so nonlocal strain gradient surface/interface effects Eqs. (7-9) and also following dimensionless parameters:

$$\begin{aligned}& \bar{u}_n = \frac{u_n}{h_{Nn}}, \bar{v}_n = \frac{v_n}{h_{Nn}}, \bar{w}_n = \frac{w_n}{h_{Nn}}, \\ & \xi_n = \frac{x_n}{L}, \bar{b}_n = \frac{b_n}{L}, \bar{A}_{ijNn} = \frac{A_{ijNn}}{A_{11Nn}},\end{aligned} \tag{29}$$

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$$\bar{B}_{ijNn}=\frac{B_{ijNn}}{A_{11Nn}h_{Nn}},$$
$$\bar{D}_{ijNn}=\frac{D_{ijNn}}{A_{11Nn}h_{Nn}^{2}},\bar{A}_{ijpn}=\frac{A_{ijpn}}{A_{11Nn}},$$
$$\bar{A}_{ijn}^{*}=\frac{A_{ijn}^{*}}{A_{11Nn}},\bar{B}_{ijpn}=\frac{B_{ijpn}}{A_{11Nn}h_{Nn}},$$
$$\bar{B}_{ijn}^{*}=\frac{B_{ijn}^{*}}{A_{11Nn}h_{Nn}},$$
$$\bar{D}_{ijpn}=\frac{D_{ijpn}}{A_{11Nn}h_{Nn}^{2}},\bar{D}_{ijn}^{*}=\frac{D_{ijn}^{*}}{A_{11Nn}h_{Nn}^{2}},$$
$$\bar{F}_{11Nn}^{*}=\frac{F_{11Nn}^{*}}{A_{11Nn}h_{Nn}},\bar{F}_{11pn}^{*}=\frac{F_{11pn}^{*}}{A_{11Nn}h_{Nn}},$$
$$\bar{E}_{11Nn}^{*}=\frac{E_{11Nn}^{*}}{A_{11Nn}h_{Nn}^{2}},\bar{E}_{11pn}^{*}=\frac{E_{11pn}^{*}}{A_{11Nn}h_{Nn}^{2}},$$
$$\bar{J}_{11Nn}^{*}=\frac{J_{11Nn}^{*}}{\rho_{Nn}h_{Nn}^{2}},J_{11pn}^{*}=\frac{J_{11pn}^{*}}{\rho_{Nn}h_{Nn}^{2}},$$
$$\bar{G}_{11Nn}^{*}=\frac{G_{11Nn}^{*}}{\rho_{Nn}h_{Nn}^{3}},G_{11pn}^{*}=\frac{G_{11pn}^{*}}{\rho_{Nn}h_{Nn}^{3}},$$
$$\bar{N}_{xpn}^{*}=\frac{N_{xpn}^{*}V_0}{A_{11Nn}},\bar{N}_{\theta pn}^{*}=\frac{N_{\theta pn}^{*}V_0}{A_{11Nn}},$$
$$\bar{M}_{xpn}^{*}=\frac{M_{xpn}^{*}V_0}{A_{11Nn}h_{Nn}},$$
$$\bar{M}_{\theta pn}^{*}=\frac{M_{\theta pn}^{*}V_0}{A_{11Nn}h_{Nn}},\bar{\tau}_0^{sn}=\frac{\tau_0^{sn}}{A_{11Nn}},$$
$$\bar{R}_n=\frac{R_n}{L},m_{0n}=\frac{L}{R_n},m_{1n}=\frac{L}{h_{Nn}},$$
$$m_{2n}=\frac{h_{Nn}}{R_n}=\bar{h}_{Nn},$$
$$\bar{h}_{pn}=\frac{h_{pn}}{R_n},m_{3n}=\frac{I_n}{2\rho_{Nn}h_{Nn}},$$
$$m_{4n}=\frac{h_{pn}}{h_{Nn}},\Omega_{\mathrm{n}}=\sqrt{\frac{A_{11Nn}}{2\rho_{Nn}h_{Nn}L^2}},$$
$$\tau_n=\Omega_{\mathrm{n}}t_n,\bar{\omega}_n=\frac{\omega_n}{\Omega_{\mathrm{n}}},$$
$$\bar{K}_w=\frac{K_wL^2}{m_3A_{11N}},\bar{K}_p=\frac{K_p}{m_3A_{11N}},$$
$$\bar{C}_{w2}=\frac{C_w\Omega L^2}{m_{3n}A_{11Nn}},\bar{\eta}=\left(\frac{l}{L}\right)^2,$$
$$\bar{\mu}=\left(\frac{e_0a}{L}\right)^2,\bar{\rho}_f=\frac{\rho_f}{m_3\rho_N},$$
$$\bar{C}_{vdwn}^{L}=\frac{C_{vdwn}^{L}L^2}{m_{3n}A_{11Nn}},$$
$$\bar{C}_{vdwn}^{NL}=\frac{C_{vdwn}^{NL}L^2h_{Nn}^2}{m_{3n}A_{11Nn}},$$
$$\bar{u}_f=\left(VCF\times V_{no-slip}\right)\sqrt{\frac{2\rho_Nh_N}{A_{11N}}},$$
$$\bar{V}_{DC}=\frac{V_{DC}}{V_0},\bar{\mu}_f=\frac{\mu_f}{m_3}\sqrt{\frac{2h_N}{\rho_NA_{11N}L^2}},$$
$$\bar{V}_{p2}=\frac{V_{p2}}{V_0},\bar{F}_e=\frac{\pi m_1^2V_0^2\Upsilon}{m_3A_{11N}},$$
$$\bar{F}=\frac{fL^2}{A_{11N}m_3{h_N}^2},$$

The dimensionless nonclassical governing equations can be expressed as:

$$(1-\bar{\eta}\,\bar{\nabla}^2)\times\begin{pmatrix}\alpha_{1un}\frac{\partial^2\bar{u}_n}{\partial\xi^2}+\alpha_{2un}\frac{\partial^2\bar{u}_n}{\partial\theta^2}\\+\alpha_{3un}\frac{\partial^2\bar{v}_n}{\partial\xi\partial\theta}+\alpha_{4un}\frac{\partial\bar{w}_n}{\partial\xi}\\+\alpha_{5un}\frac{\partial\bar{w}_n}{\partial\xi}\frac{\partial^2\bar{w}_n}{\partial\xi^2}+\\\alpha_{6un}\frac{\partial\bar{w}_n}{\partial\xi}\frac{\partial^2\bar{w}_n}{\partial\theta^2}+\alpha_{7un}\frac{\partial^2\bar{w}_n}{\partial\xi\partial\theta}\frac{\partial\bar{w}_n}{\partial\theta}\end{pmatrix}=(1-\bar{\mu}\,\bar{\nabla}^2)\frac{\partial^2\bar{u}_n}{\partial\tau^2},\tag{30}$$

$$(1-\bar{\eta}\,\bar{\nabla}^2)\times\begin{pmatrix}\alpha_{1vn}\frac{\partial^2\bar{u}_n}{\partial\xi\partial\theta}+\alpha_{2vn}\frac{\partial^2\bar{v}_n}{\partial\xi^2}+\\\alpha_{3vn}\frac{\partial^2\bar{v}_n}{\partial\theta^2}+\alpha_{4vn}\frac{\partial\bar{w}_n}{\partial\xi}\frac{\partial^2\bar{w}_n}{\partial\xi\partial\theta}\\+\alpha_{5vn}\frac{\partial^2\bar{w}_n}{\partial\xi^2}\frac{\partial\bar{w}_n}{\partial\theta}+\alpha_{6vn}\frac{\partial\bar{w}_n}{\partial\theta}+\\\alpha_{7vn}\frac{\partial\bar{w}_n}{\partial\theta}\frac{\partial^2\bar{w}_n}{\partial\theta^2}\end{pmatrix}=(1-\bar{\mu}\,\bar{\nabla}^2)\frac{\partial^2\bar{v}_n}{\partial\tau^2},\tag{31}$$

$$(1-\bar{\eta}\,\bar{\nabla}^2)\times\tag{32}$$

$$\begin{pmatrix} \alpha_{1wn}\frac{\partial \bar{u}_n}{\partial \xi} + \alpha_{2wn}\frac{\partial \bar{u}_n}{\partial \xi}\frac{\partial^2 \bar{w}_n}{\partial \xi^2} + \\ \alpha_{3wn}\frac{\partial \bar{u}_n}{\partial \xi}\frac{\partial^2 \bar{w}_n}{\partial \theta^2} + \alpha_{4wn}\frac{\partial^2 \bar{u}_n}{\partial \xi^2}\frac{\partial \bar{w}_n}{\partial \xi} \\ + \alpha_{5wn}\frac{\partial^2 \bar{u}_n}{\partial \xi \partial \theta}\frac{\partial \bar{w}_n}{\partial \theta} + \alpha_{6wn}\frac{\partial^2 \bar{u}_n}{\partial \xi \partial \theta}\frac{\partial \bar{w}_n}{\partial \theta} + \\ \alpha_{7wn}\frac{\partial \bar{u}_n}{\partial \theta}\frac{\partial^2 \bar{w}}{\partial \xi \partial \theta} + \alpha_{8wn}\frac{\partial^2 \bar{u}_n}{\partial \theta^2}\frac{\partial \bar{w}_n}{\partial \xi} \\ +\alpha_{9wn}\frac{\partial \bar{v}_n}{\partial \xi}\frac{\partial^2 \bar{w}_n}{\partial \xi \partial \theta} + \alpha_{10wn}\frac{\partial^2 \bar{v}_n}{\partial \xi^2}\frac{\partial \bar{w}_n}{\partial \theta} + \\ \alpha_{11wn}\frac{\partial^2 \bar{v}_n}{\partial \xi \partial \theta}\frac{\partial \bar{w}_n}{\partial \xi} + \alpha_{12wn}\frac{\partial \bar{v}_n}{\partial \theta} \\ +\alpha_{13wn}\frac{\partial \bar{v}_n}{\partial \theta}\frac{\partial^2 \bar{w}_n}{\partial \xi^2} + \alpha_{14wn}\frac{\partial \bar{v}_n}{\partial \theta}\frac{\partial^2 \bar{w}_n}{\partial \theta^2} \\ +\alpha_{15wn}\frac{\partial^2 \bar{v}_n}{\partial \theta^2}\frac{\partial \bar{w}_n}{\partial \theta} + \alpha_{16wn}\bar{w}_n \\ +\alpha_{17wn}\bar{w}_n\frac{\partial^2 \bar{w}_n}{\partial \xi^2} + \alpha_{18wn}\bar{w}_n\frac{\partial^2 \bar{w}_n}{\partial \theta^2} + \\ \alpha_{19wn}\frac{\partial^2 \bar{w}_n}{\partial \xi^2} + \alpha_{20wn}\frac{\partial \bar{w}_n}{\partial \xi}\frac{\partial^2 \bar{w}_n}{\partial \xi \partial \theta}\frac{\partial \bar{w}_n}{\partial \theta} \\ +\alpha_{21wn}\frac{\partial^2 \bar{w}_n}{\partial \xi^2}\left(\frac{\partial \bar{w}_n}{\partial \xi}\right)^2 + \\ \alpha_{22wn}\frac{\partial^2 \bar{w}_n}{\partial \xi^2}\left(\frac{\partial \bar{w}_n}{\partial \theta}\right)^2 + \alpha_{23wn}\frac{\partial^4 \bar{w}_n}{\partial \xi^4} \\ +\alpha_{24wn}\left(\frac{\partial \bar{w}_n}{\partial \xi}\right)^2 + \\ \alpha_{26wn}\left(\frac{\partial \bar{w}_n}{\partial \xi}\right)^2\frac{\partial^2 \bar{w}_n}{\partial \theta^2} + \alpha_{27wn}\frac{\partial^2 \bar{w}_n}{\partial \theta^2} + \\ \alpha_{28wn}\frac{\partial^4 \bar{w}_n}{\partial \theta^4} \\ +\alpha_{29wn}\frac{\partial^2 \bar{w}_n}{\partial \theta^2}\left(\frac{\partial \bar{w}_n}{\partial \theta}\right)^2 + \\ \alpha_{30wn}\left(\frac{\partial \bar{w}_n}{\partial \theta}\right)^2 + \alpha_{31wn}\frac{\partial^4 \bar{w}_n}{\partial \xi^2 \partial \tau^2} + \\ \alpha_{32wn}\frac{\partial^4 \bar{w}_n}{\partial \theta^2 \partial \tau^2} \\ +\alpha_{33wn} \end{pmatrix}$$

$$= (1-\bar{\mu}\,\bar{\nabla}^2)\times \begin{pmatrix} \frac{\partial^2 \bar{w}_n}{\partial \tau^2} + \bar{S}_n - \bar{F}cos\bar{\Omega}\tau \\ -\frac{\bar{F}_{e2}\bar{V}_{DC2}^2}{\sqrt{\begin{pmatrix}(m_{12}\bar{b}_2 - \bar{w}_2)\times \\ (2m_{42}\bar{R}_2 + m_{12}\bar{b}_2 - \bar{w}_2)\end{pmatrix}}\times \\ \left[cosh^{-1}\left(1+\frac{m_{12}\bar{b}_2 - \bar{w}_2}{m_{42}\bar{R}_2}\right)\right]^2} \end{pmatrix},$$

where $\bar{\nabla}^2 = \frac{\partial^2}{\partial \xi^2} + m_0^2\frac{\partial^2}{\partial \theta^2}$ and $\bar{S}_n$ for three layers, respectively, are:

$$\bar{S}_1 = -\begin{pmatrix}\bar{C}_{vdw(12)}^{L}(\bar{w}_2 - \bar{w}_1) + \\ \bar{C}_{vdw(12)}^{NL}(\bar{w}_2 - \bar{w}_1)^3\end{pmatrix},) \quad (33)$$

$$\bar{S}_2 = \begin{pmatrix} -\bar{C}_{vdw(23)}^{L}(\bar{w}_3 - \bar{w}_2) - \\ \bar{C}_{vdw(23)}^{NL}(\bar{w}_3 - \bar{w}_2)^3 \\ +\left(\frac{\bar{R}_1}{\bar{R}_2}\right)\begin{pmatrix}C_{vdw(12)}^{L}(\bar{w}_2 - \bar{w}_1) + \\ \bar{C}_{vdw(12)}^{NL}(\bar{w}_2 - \bar{w}_1)^3\end{pmatrix}\end{pmatrix} \quad (34)$$

$$\bar{S}_3 = \begin{pmatrix}\left(\frac{\bar{R}_2}{\bar{R}_3}\right)\begin{pmatrix}\bar{C}_{vdw(32)}^{L}(\bar{w}_3 - \bar{w}_2) + \\ \bar{C}_{vdw(32)}^{NL}(\bar{w}_3 - \bar{w}_2)^3\end{pmatrix} + \\ \bar{C}_w\frac{\partial \bar{w}_3}{\partial \tau}\end{pmatrix} \quad (35)$$

All coefficients of $\alpha_{iu}(i = 1..7), \alpha_{jv}(j = 1..7)$ and $\alpha_{kw}(k = 1..33)$ are introdacted in Appendix 1. The associated boundary conditions can be obtained in the dimensionless form as:

$$\delta\bar{u}_n = 0: \begin{pmatrix}\alpha_{1un}^{bc}\frac{\partial \bar{u}_n}{\partial \xi} + \alpha_{2un}^{bc}\frac{\partial \bar{v}_n}{\partial \theta} + \\ \alpha_{3un}^{bc}\bar{w}_n + \alpha_{4un}^{bc}\left(\frac{\partial \bar{w}_n}{\partial \xi}\right)^2 + \\ \alpha_{5un}^{bc}\left(\frac{\partial \bar{w}_n}{\partial \theta}\right)^2 + \alpha_{6un}^{bc}\end{pmatrix}\delta\bar{u}_{\xi n}\Big|_0^l + \begin{pmatrix}\alpha_{7un}^{bc}\frac{\partial \bar{u}_n}{\partial \theta} + \alpha_{8un}^{bc}\frac{\partial \bar{v}_n}{\partial \xi} + \\ \alpha_{9un}^{bc}\frac{\partial \bar{w}_n}{\partial \xi}\frac{\partial \bar{w}_n}{\partial \theta}\end{pmatrix}\delta\bar{u}_{\theta n}\Big|_0^{2\pi} = 0, \quad (36)$$

$$\delta\bar{v}_n = 0: \begin{pmatrix}\alpha_{1vn}^{bc}\frac{\partial \bar{u}_n}{\partial \theta} + \alpha_{2vn}^{bc}\frac{\partial \bar{v}_n}{\partial \xi} + \\ \alpha_{3vn}^{bc}\frac{\partial \bar{w}_n}{\partial \xi}\frac{\partial \bar{w}_n}{\partial \theta}\end{pmatrix}\delta\bar{v}_{\xi n}\Big|_0^l + \begin{pmatrix}\alpha_{4vn}^{bc}\frac{\partial \bar{u}_n}{\partial \xi} + \alpha_{5vn}^{bc}\frac{\partial \bar{v}_n}{\partial \theta} + \alpha_{6vn}^{bc}\bar{w}_n + \\ \alpha_{7vn}^{bc}\left(\frac{\partial \bar{w}_n}{\partial \xi}\right)^2 + \alpha_{8vn}^{bc}\left(\frac{\partial \bar{w}_n}{\partial \theta}\right)^2 + \alpha_{9vn}^{bc}\end{pmatrix}\delta\bar{v}_{\theta n}\Big|_0^{2\pi} = 0, \quad (37)$$

$$\delta\bar{w}_n = 0: \quad (38)$$

$$\left(\begin{array}{c} +\alpha_{1wn}^{bc}\frac{\partial\bar{u}_n}{\partial\xi}\frac{\partial\bar{w}_n}{\partial\xi}+\alpha_{2wn}^{bc}\frac{\partial\bar{u}_n}{\partial\theta}\frac{\partial\bar{w}_n}{\partial\theta}+ \\ \alpha_{3wn}^{bc}\frac{\partial\bar{v}_n}{\partial\xi}\frac{\partial\bar{w}_n}{\partial\theta}+\alpha_{4wn}^{bc}\frac{\partial\bar{v}_n}{\partial\theta}\frac{\partial\bar{w}_n}{\partial\xi}+ \\ \alpha_{5wn}^{bc}\bar{w}_n\frac{\partial\bar{w}_n}{\partial\xi} \\ +\alpha_{6wn}^{bc}\frac{\partial\bar{w}_n}{\partial\xi}+\alpha_{7wn}^{bc}\frac{\partial^3\bar{w}_n}{\partial\xi^3}+ \\ \alpha_{8wn}^{bc}\frac{\partial^3\bar{w}_n}{\partial\xi\partial\theta^2}+ \\ \alpha_{9wn}^{bc}\frac{\partial\bar{w}_n}{\partial\xi}\left(\frac{\partial\bar{w}_n}{\partial\xi}\right)^2 \\ +\alpha_{10wn}^{bc}\frac{\partial\bar{w}_n}{\partial\xi}\left(\frac{\partial\bar{w}_n}{\partial\theta}\right)^2+\alpha_{11wn}^{bc}\frac{\partial^3\bar{w}_n}{\partial\xi\partial\tau^2}+ \\ \alpha_{12wn}^{bc}\frac{\partial\bar{u}_n}{\partial\xi}\frac{\partial\bar{w}_n}{\partial\theta}+\alpha_{13wn}^{bc}\frac{\partial\bar{u}_n}{\partial\theta}\frac{\partial\bar{w}_n}{\partial\xi} \end{array}\right)\delta\bar{w}_{\xi n}\Big|_0^l$$

$$+\left(\begin{array}{c} +\alpha_{14wn}^{bc}\frac{\partial\bar{v}_n}{\partial\xi}\frac{\partial\bar{w}_n}{\partial\xi}+\alpha_{15wn}^{bc}\frac{\partial\bar{v}_n}{\partial\theta}\frac{\partial\bar{w}_n}{\partial\theta} \\ +\alpha_{16wn}^{bc}\bar{w}_n\frac{\partial\bar{w}_n}{\partial\theta}+\alpha_{17wn}^{bc}\frac{\partial^3\bar{w}_n}{\partial\xi^2\partial\theta} \\ +\alpha_{18wn}^{bc}\left(\frac{\partial\bar{w}_n}{\partial\xi}\right)^2\frac{\partial\bar{w}_n}{\partial\theta} \\ +\alpha_{19wn}^{bc}\frac{\partial\bar{w}_n}{\partial\theta}+ \\ \alpha_{20wn}^{bc}\frac{\partial^3\bar{w}_n}{\partial\theta^3}+\alpha_{21wn}^{bc}\frac{\partial\bar{w}_n}{\partial\theta}\left(\frac{\partial\bar{w}_n}{\partial\theta}\right)^2 \\ +\alpha_{22wn}^{bc} \end{array}\right)\delta\bar{w}_{\theta n}\Big|_0^{2\pi}$$

$$=0,$$

$$\frac{\partial\bar{w}_n}{\partial\xi}=0:$$

$$\left(\begin{array}{c} \alpha_{23wn}^{bc}\frac{\partial^2\bar{w}_n}{\partial\xi^2}+\alpha_{24wn}^{bc}\frac{\partial^2\bar{w}_n}{\partial\theta^2} \\ +\alpha_{25wn}^{bc}+\alpha_{26wn}^{bc}\frac{\partial^2\bar{w}_n}{\partial\xi^2} \\ +\alpha_{27wn}^{bc}\frac{\partial^2\bar{w}_n}{\partial\theta^2} \\ +\alpha_{28wn}^{bc}\frac{\partial^2\bar{w}_n}{\partial\tau^2} \end{array}\right)\delta\left(\frac{\partial\bar{w}_{\xi n}}{\partial\xi}\right)\Big|_0^l$$

$$+\left(\alpha_{29wn}^{bc}\frac{\partial^2\bar{w}_n}{\partial\xi\partial\theta}\right)\delta\left(\frac{\partial\bar{w}_{\theta n}}{\partial\xi}\right)\Big|_0^{2\pi}\quad=0,$$

$$\frac{\partial\bar{w}_n}{\partial\theta}=0:\left(\alpha_{30wn}^{bc}\frac{\partial^2\bar{w}_n}{\partial\xi\partial\theta}\right)\delta\left(\frac{\partial\bar{w}_{\xi n}}{\partial\theta}\right)\Big|_0^l$$

$$+\left(\begin{array}{c} \alpha_{31wn}^{bc}\frac{\partial^2\bar{w}_n}{\partial\xi^2}+\alpha_{32wn}^{bc}\frac{\partial^2\bar{w}_n}{\partial\theta^2}+ \\ \alpha_{33wn}^{bc}\alpha_{34wn}^{bc}\frac{\partial^2\bar{w}_n}{\partial\xi^2}+\alpha_{35wn}^{bc}\frac{\partial^2\bar{w}_n}{\partial\theta^2} \\ ++\alpha_{36wn}^{bc}\frac{\partial^2\bar{w}_n}{\partial\tau^2} \end{array}\right)\delta\left(\frac{\partial\bar{w}_{\theta n}}{\partial\theta}\right)\Big|_0^{2\pi}$$

$$=0,$$

where coefficients of $\alpha_{lun}^{bc}(l=1..9)$, $\alpha_{mvn}^{bc}(m=1..9)$ and $\alpha_{nwn}^{bc}(n=1..36)$ are introduced in Appendix 2. In current study, the electrostatic force Eq. (9) can be expressed as a polynomial form that is solved by nonlinear curve-fitting problem of lsqcurvefit function in Matlab Toolbox using least-squares. Therefore, the dimensionless electrostatic work can be written as follows (Hashemi Kachapi *et al.* 2019c):

$$W_e=\int_0^L\int_0^{2\pi}\int_0^{\bar{w}_3}\begin{array}{c}\bar{F}_e(\bar{V}_{DC}+\bar{V}_{AC}\cos(\bar{\omega}\tau))^2 \\ \times\left(\begin{array}{c}\bar{C}_1+\bar{C}_2\bar{w}_3+ \\ \bar{C}_3\bar{w}_3^2+\cdots \\ +\bar{C}_n\bar{w}_3^{n-1}\end{array}\right)\end{array}\times d\bar{w}_3\bar{R}_3\,d\theta\,d\xi \tag{41}$$

which $\bar{C}_1-\bar{C}_n$ are constant.

2.3 Solution procedure

In this work, the assumed mode method is used to obtain the equations of motion using Lagrange– Euler method. In the assumed mode method displacements are written in terms of generalized coordinate and mode function as follows (Amabili [29])

$$\begin{bmatrix}u_n(x,\theta,t)\\ v_n(x,\theta,t)\\ w_n(x,\theta,t)\end{bmatrix}=\sum_{m=1}^{M_1}\sum_{j=1}^{N}\begin{bmatrix}\begin{bmatrix}u_{m,j,c}(\tau)\cos(j\theta)+\\ u_{m,j,s}(\tau)\sin(j\theta)\end{bmatrix}\chi_{mj}(\xi)\\ \begin{bmatrix}v_{m,j,c}(\tau)\sin(j\theta)+\\ v_{m,j,s}(\tau)\cos(j\theta)\end{bmatrix}\phi_{mj}(\xi)\\ \begin{bmatrix}w_{m,j,c}(\tau)\cos(j\theta)+\\ w_{m,j,s}(\tau)\sin(j\theta)\end{bmatrix}\beta_{mj}(\xi)\end{bmatrix}+\sum_{m=1}^{M_2}\begin{bmatrix}u_{m,0}(\tau)\chi_{m0}(\xi)\\ v_{m,0}(\tau)\phi_{m0}(\xi)\\ w_{m,0}(\tau)\beta_{m0}(\xi)\end{bmatrix}=\sum_{(i,r,s)=1}^{M_2+M_1\times N}\begin{bmatrix}u_{ni}(\tau)\chi_{ni}(\xi)\vartheta_{ni}(\theta)\\ v_{nr}(\tau)\phi_{nr}(\xi)\alpha_{nr}(\theta)\\ w_{ns}(\tau)\beta_{ns}(\xi)\psi_{ns}(\theta)\end{bmatrix}, \tag{42}$$

where $\chi_{ni}(\xi)$, $\phi_{nr}(\xi)$ and $\beta_{ns}(\xi)$ are modal functions which satisfy the required geometric boundary conditions. $u_{ni}(\tau)$, $v_{nr}(\tau)$ and $w_{ns}(\tau)$

are unknown functions of time and are related to dynamical response.

By substituting Eqs. (42) into Eqs. (30-32) and (36-40)) and applying the Galerkin technique, the reduced-order equation of motion is written to the following form:

$$[(K)_u^u + (K_{bc})_u^u]_n\{\bar{u}_n\} + [(K)_u^v + (K_{bc})_u^v]_n\{\overline{v_n}\} + [(K)_u^w + (K_{bc})_u^w]_n\{\bar{w}_n\} + [(NL)_u^w + (NL_{bc})_u^w]_n\{\bar{w}_n^2\} = [(M)_u^u]_n\{\ddot{\bar{u}}_n\} + \bar{F}_{upn}^{bc}, \tag{43}$$

$$[(K)_v^u + (K_{bc})_v^u]_n\{\bar{u}_n\} + [(K)_v^v + (K_{bc})_v^v]_n\{\bar{v}_n\} + [(K)_v^w + (K_{bc})_v^w]_n\{\bar{w}_n\} + [(NL)_v^w + (NL_{bc})_v^w]_n\{\bar{w}_n^2\} = [(M)_v^v]_n\{\ddot{\bar{v}}_n\} + \bar{F}_{vpn}^{bc}, \tag{44}$$

$$\begin{aligned}
&[(K)_w^u]_n\{\bar{u}_n\} + [(K)_w^v]_n\{\bar{v}_n\} + \left[(K)_w^w + (K_{bc})_w^w - \left(K_{vp}\right)_w^w - (K_{e2})_w^w\right]_n\{\bar{w}_n\} \\
&+[(NL)_w^u + (NL_{bc})_w^u]_n\{\bar{w}_n\bar{u}_n\} + [(NL)_w^v + (NL_{bc})_w^v]_n\{\bar{w}_n\bar{v}_n\} \\
&+[(NL)_{w2}^w + (NL_{bc})_{w2}^w - (NL_{2e})_{w2}^w]_n\{\bar{w}_n^2\} \\
&+[(NL)_{w3}^w + (NL_{bc})_{w3}^w - (NL_{3e})_{w3}^w]_n\{\bar{w}_n^3\} \\
&= \\
&([(M)_w^w + (M_{bc})_w^w]_n)\{\ddot{\bar{w}}_n\} + ([(C)_w^w] + [(C_{bc})_w^w]_n)\{\dot{\bar{w}}_n\}) \\
&+(-1)^n q_1\left(\frac{\bar{R}_{n-1}}{\bar{R}_n}\right)^{m_1} \bar{C}_{vdw((n-1)n)}^{\,L}\left(\left[(K)_{w1n}^{vdw}\right] - \left[(K)_{w2n}^{vdw}\right]\{\bar{w}_{n-1}\}\right) \\
&+(-1)^n q_2\left(\frac{\bar{R}_n}{\bar{R}_{n+1}}\right)^{m_2} \bar{C}_{vdw(n(n+1))}^{\,L}\left(\left[(K)_{w3n}^{vdw}\right] - \left[(K)_{w4n}^{vdw}\right]\{\bar{w}_n\}\right) \\
&+(-1)^n q_1\left(\frac{\bar{R}_{n-1}}{\bar{R}_n}\right)^{m_1}\left(\bar{C}_{vdw((n-1)n)}^{\,NL}\right)\begin{pmatrix} [(N \\ +3[(\end{pmatrix} \\
&+(-1)^n q_2\left(\frac{\bar{R}_n}{\bar{R}_{n+1}}\right)^{m_2}\left(\bar{C}_{vdw(n(n+1))}^{\,NL}\right)\begin{pmatrix} [(NL) \\ +3[(\end{pmatrix} \\
&+\bar{F}_{wpn} + \bar{F}_{wpn}^{bc} - \bar{F}_{we2} - [\bar{F}cos\bar{\Omega}\tau] \\
&-\bar{F}_e\{((\bar{V}_{AC}cos\bar{\omega}\tau)^2 + 2\bar{V}_{AC}\bar{V}_{DC}cos\bar{\omega}\tau)(\bar{C}_4(NL_{3e})_w^w + \bar{C}_3(NL_{2e})_w^w + \bar{C}_2(K_e)_w^w + \bar{C}_1\bar{F}_1)\},
\end{aligned} \tag{45}$$

where $[(M)]_n$, $[(C)]_n$ and $[(K)]_n$ are mass, damping and stiffness matrixes, respectively, in directions of u_n, v_n and w_n. $[(NL)_u^w]_n$, $[(NL)_v^w]_n$, and $[(NL)_{w2}^w]_n$ are second-order nonlinear stiffness matrixes and $[(NL)_{w3}^w]_n$ is third-order nonlinear stiffness matrix. Also,$[K_e]$, $[NL_{2e}]$ and $[NL_{3e}]$ are the linear stiffness, second and third order nonlinear stiffness matrixes for electrostatic force expansion, respectively. Also, $\bar{F}_{un}$, $\bar{F}_{vn}$ and $\bar{F}_{wn}$ are applied loads by piezoelectric voltage and surface stress. $[(K)_w^{vdw}]_n$ is stiffness matrix for van der Walls effect,

for $n = 1$: $m_2 = q_1 = 0$; $q_2 = 1$; for $n = 2$: $m_2 = 0$; $m_1 = q_1 = 1$; $q_2 = -1$;and for $n = 3$: $q_2 = 0$; $m_1 = 1$; $q_1 = -1$. All coefficients of Eqs. (43)-(45) are presented in Appendix 3.

III. RESULTS AND DISCUSSIONS

In this section, comparison of nonclassical theories i.e. NLT, NSGT and GMSIT with classical theory CT is investigated. For this purpose, the effect of different material and geometrical parameters with and without nonlocal, nonlocal strain gradient, surface/interface effects and also pull-in voltage $\bar{V}_{DC}$ will be discussed on dimensionless natural frequency (DNF) with specifications mentioned to Tables 1-3. In order to simplify the presentation, CC, SS, CS and CF respectively represent clamped edge, simply supported edge, clamped-simply supported edge and clamped-free edge. The material properties for the different layers of nanoshell (Al), piezoelectric layer (PZT-4) and others geometrical parameters for bulk and surface of MWPENS are shown in Tables 1-3, respectively (Ghorbanpour Arani et al. [28], Rouhi *et al.* [33], Hashemi Kachapi *et al.* [17, 20, 25]).

Table 1 Surface and bulk properties of Al

$E_N(GI$	υ_N	$\rho_N(k$ $m^3)$	$\lambda^I(N/$	$\mu^I(N/$	$\tau_0^I(N/$	$\rho^I(kg/$
70	0.33	2700	3.786	1.95	0.9108	$5.46 \times$

Table 2 Surface and bulk properties of PZT-4

$C_{11p}(G$	$C_{22p}(G$	$C_{12p}(GPa$	$C_{21p}(G$	$C_{66p}(G$	$E_p(GI$
139	139	77.8	77.8	30.5	95
υ_p	$\rho_p(kg$	$\eta_{33p}(10^{-}$	$\lambda^S(N/$	$\mu^S(N/$	$\tau_0^S(N$
0.3	7500	8.91	4.488	2.774	0.6048
$e_{31p}(C$	$e_{32p}(C$	$e_{31p}^S(C/$	$e_{32p}^S(C$	$\rho^S(kg/$	
-5.2	-5.2	$-3 \times 10^{-}$	-3×1	$5.61 \times$	

Table 3 The material and geometrical parameters

$R_1(m)$	$R_2(m)$	$R_2(m)$	L/R_1	$h_{N_{(1,2,3)}}/R_1$
1×10^{-9}	$1.5 \times 10^{-}$	2×10	10	0.01
h_{p_3}/R_1	b_3/R_3	V_{03}	$V_{p_3}(V)$	$V_{DC_3}(V)$
0.005	0.1	1	1×10	5
$C_{vdw_3}^L(N/$	$C_{vdw_3}^{NL}(N/$	$F(N)$	$\bar{\mu}$	$\eta(m^2)$
9.918666×10^{19}	2.201667×10^{31}	50	0.05	$(0.01 \times 10$

Of course, the geometrical parameters can be varying according to the type of problem. In this work, the results are presented in dimensionless form and thus the results are not limited to a particular type of matter. The data presented in the form of sample data to approximate the values used in the actual range.

A. Verification and comparison

In this section, a comparison is made between the dimensionless natural frequencies of SS nanoshell obtained in this works, with the results presented in Rouhi *et al.* [33] and Ansari *et al.* [34] with material and geometrical parameters of: $E = 210\,GP$, $v = 0.24$, $\rho = 2331\,kg\,m^{-3}$, $\lambda^s = -4.488\,N\,m^{-1}$, $\mu^s = -2.774\,N\,m^{-1}$, $\tau_0^s = 0.605\,N\,m^{-1}$, $\rho_s = 3.17 \times 10^{-7}\,kg\,m^{-2}$, $h_N = 1\,nm$, $R\,/h_N = 2.5$, and $(m,n) = (3,3)$. It can be observed from Table 4 that the present results agree very well with the results mentioned in references, and indicates that the method presented in this work has sufficient accuracy and the insignificant differences in the results is due to the different shell theories and solution methodology used in the literature and in this work.

Table 4 Comparison of dimensionless natural frequencies for SS boundary condition

L/R	Present	Surface stress shell model (Rouhi *et al.* 2015)	Surface stress beam model (Ansari *et al.* 2015)
45	0.2288	0.2363	0.2341
90	0.2138	0.2208	0.2204
135	0.2109	0.2137	0.2126
200	0.2097	0.2083	0.2076

B. Effects of nonclassical approach on DNF and pull-in voltage

The main purpose of this section is to compare three nonclassical theories of NLT, NSGT and GMSIT with classical theory CT to investigate effects of surface/interface, nonlocal scale and, material length scale parameters on pull-in voltage $\bar{V}_{DC}$ and dimensionless natural frequency (DNF). It is noted that in all following results, in NLT only consider $\bar{\mu}$ effect; in NSGT only consider $\bar{\mu}$ and $\bar{\eta}$ effects; in GMSIT only consider all S/I effects and CT not consider $\bar{\mu}$, $\bar{\eta}$ and all S/I effects. In following, first dimensionless natural frequencies of MWPENS versus dimensionless nonlocal scale parameter $\bar{\mu}$, material length scale parameter $\bar{\eta}$ and pull-in voltage $\bar{V}_{DC}$ with and without surface/interface effects for different boundary conditions respectively are presented in Figures 2-4.

It is clear from the Figures 2-4 that in the case of with surface/interface effect, due to increasing of MWPENS stiffness, the dimensionless natural frequency increase compared to case of without S/I effects.

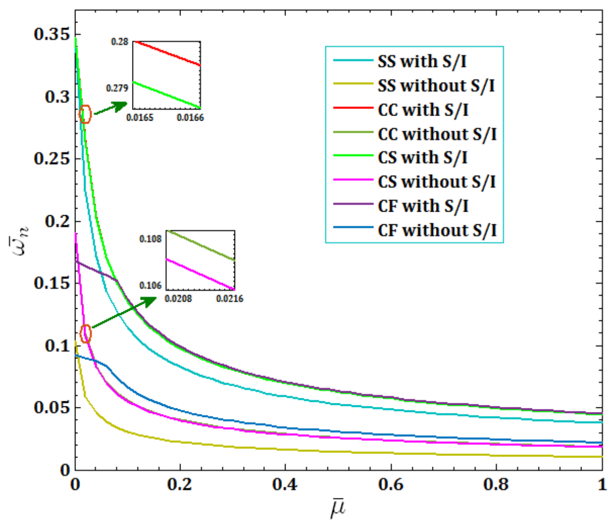

Fig. 2 The surface/interface effects on natural frequency versus nonlocal scale parameter $\bar{\mu}$ with $\bar{\eta} = 0.01$ for different boundary conditions

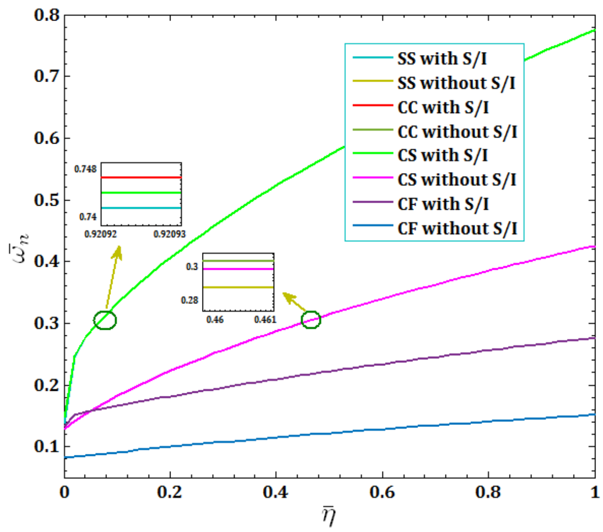

Fig. 3 The surface/interface effects on natural frequency versus material length scale parameter $\bar{\eta}$ with $\bar{\mu} = 0.1$ for different boundary conditions

Corresponding to Figure 2, in all cases, with increasing nonlocal parameter, the DNF decreases. Also, it is clear that in all cases of Figure 3, due to increasing of MWPENS stiffness, the DNF increases with increasing dimensionless material length scale parameter $\bar{\eta}$. Increasing of pull-in voltage DC lead to decreasing of DNF. Results of Figures 2-4 indicate that nonlocal scale parameter $\bar{\mu}$, material length scale parameter $\bar{\eta}$ and pull-in voltage DC respectively lead to increasing and decreasing of MWPENS stiffness and results of lead to increasing and decreasing the DNF of MWPENS. Furthermore, in all cases of nonlocal, material length scale parameters and voltage DC, the dimensionless natural frequency related to CC boundary condition is higher than that related to CS, SS and CF one. This is due to the fact that the CC boundary condition is stiffer than other boundary conditions.

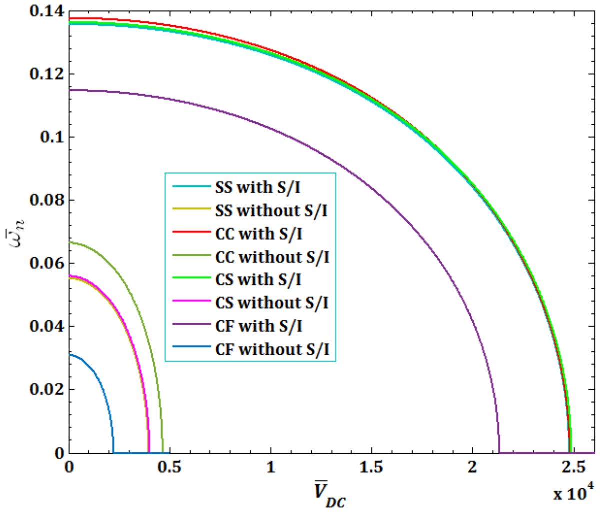

Fig. 4 The surface/interface effects on natural frequency versus pull-in voltage $\bar{V}_{DC}$ with $\bar{\mu} = 0.1$ and $\bar{\eta} = 0.01$ for different boundary conditions

Also, for zero natural frequency in Figure4, MWPENS becomes unstable and this physically implies that first the MWPENS losses its stability due to the divergence via a pitchfork bifurcation. In all following results are used values of nonlocal scale parameter $\bar{\mu} = 0.1$, material length scale parameter $\bar{\eta} = 0.01$.

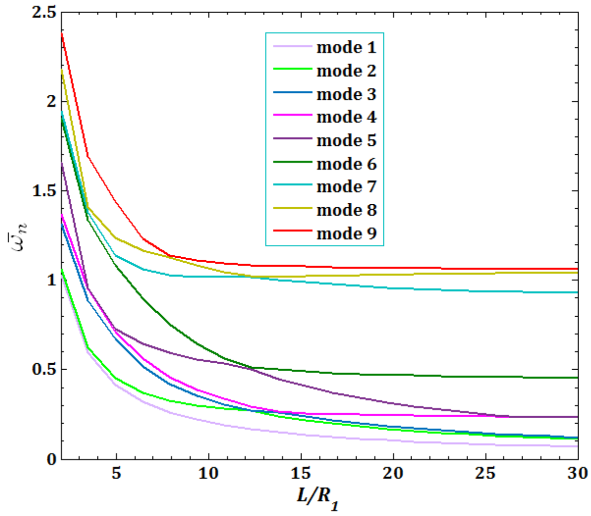

Fig.5 The effect of vibrational modes number on DNF versus SS MWPENS L/R_1 ratio considering with all NLT, NSGT and S/I theories

The relationship between the dimensionless natural frequency DNF versus different length to radius ratio L/R_1 of SS MWPENS is shown in Figure 5 for nine vibrational modes number considering with all NLT, NSGT and S/I effects. It is observed that for all modes, due to decreasing of MWPENS stiffness, the DNF decreases when the L/R_1 ratio increases. Also, the DNF for mode number 9 is higher than that for other modes. Actually with increasing of vibrational modes number, DNF increases.

Figure 6 show the comparison of three nonclassical theories of NLT, NSGT and GMSIT with classical theory CT on dimensionless natural frequency versus ratio L/R_1 for SS MWPENS. It can be seen from this Figure that the highest frequency is related to the GMSIT and shows that in this case, the system's rigidity is higher than other cases. The lowest frequency is also related to nonlocal theory NL and shows that in this case, the system hardness is less than other cases. Also, the natural frequency of the classical theory is greater than the NLT, NSGT and combination of GMSIT+NLT and GMSIT+NSGT theories, indicating a reduction in the rigidity of the system due to the consideration of these theories. This results shown that S/I effects lead to increasing of MWPENS stiffness and other nonclassical effects lead to decreasing of MWPENS stiffness and as a result decreasing of DNF.

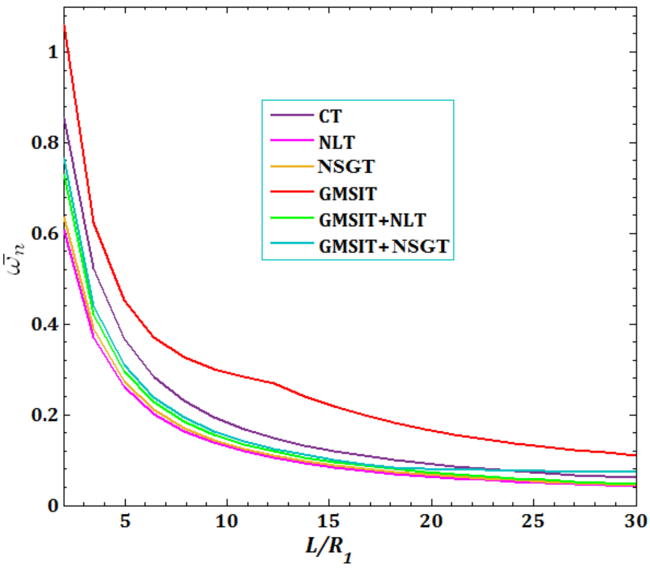

Fig.6 Comparison of nonclassical theories with classical theory on dimensionless natural frequency versus SS MWPENS L/R_1 ratio

Figure 7 shows the comparison of three nonclassical theories of NLT, NSGT and GMSIT with classical theory CT on dimensionless natural frequency versus different piezoelectric actuation voltage $\bar{V}_p$ for SS MWPENS.

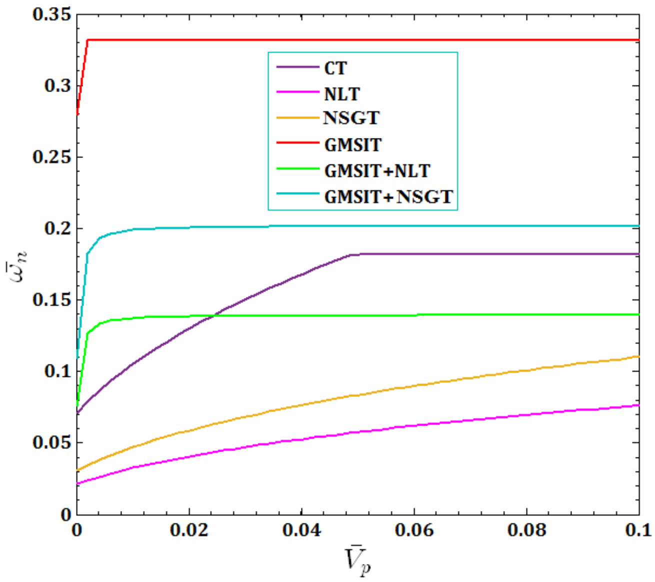

Fig.7 Comparison of nonclassical theories with classical theory on dimensionless natural frequency versus different piezoelectric voltage $\bar{V}_p$

All of the results presented in the previous Figure on the effects of various theories on natural frequencies are clearly seen in this Figure and confirm the previous results. In all case, with increasing of piezoelectric voltage and as a result increasing the rigidity of the system, the frequency increases. In theories of CT, GMSIT, GMSIT+NLT GMSIT+NSGT, the frequency increases and by most increasing of $\bar{V}_p$, piezoelectric voltage variations do not have a significant effect on the natural frequency of the MWPENS. But in NLT and NSGT, with increasing of $\bar{V}_p$, DNF increases with slightly slop and by most increasing of $\bar{V}_p$, this increasing of DNF considerable. The compare nonclassical approaches with classical theory on the natural frequency versus direct pull-in voltage DC of SS MWPENS is presented in Figure 8.

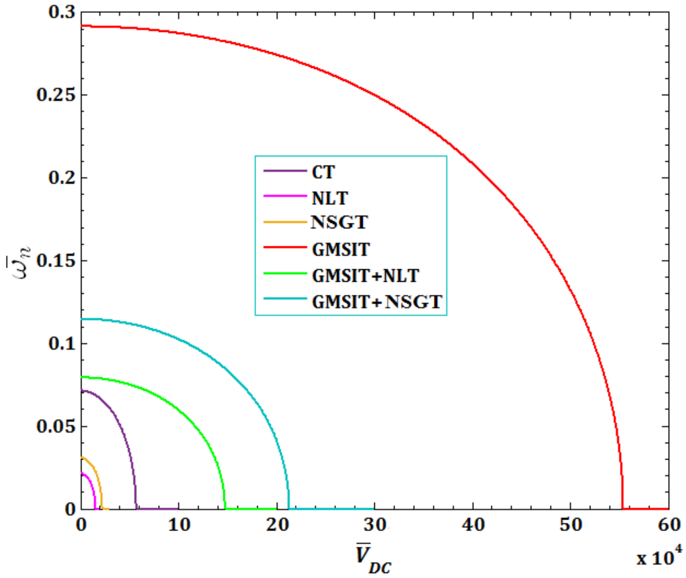

Fig. 8 Comparison of nonclassical approaches with classical theory on dimensionless natural frequency versus different pull-in voltage $\bar{V}_{DC}$ of SS MWPENS

Figures 9-11 are presented the effects of surface/interface parameters on dimensionless natural frequency respectively versus nonlocal scale parameter $\bar{\mu}$, material length scale parameter $\bar{\eta}$ and pull-in voltage $\bar{V}_{DC}$ for SS MWPENS.

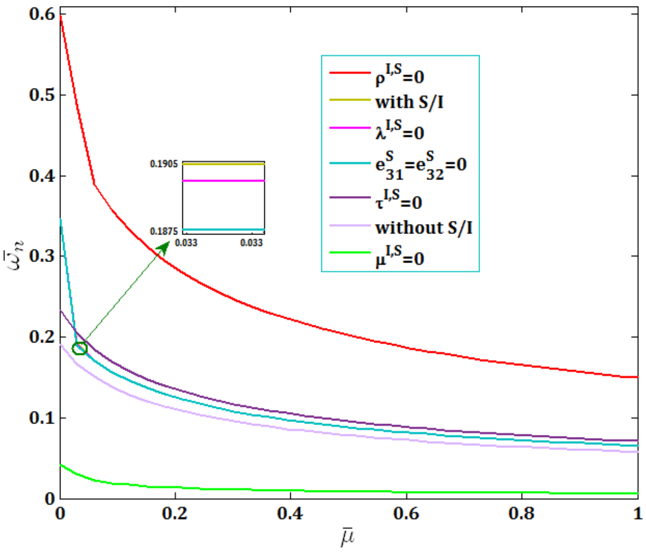

Fig. 9 The effects surface/interface parameters on DNF versus $\bar{\mu}$ with $\bar{\eta} = 0.01$ for SS MWPENS

In all three Figures, it can be seen that with ignoring the surface/interface density $\rho^{I,S}$, the inertia of the system will greatly decrease and due to increasing of SS MWPENS stiffness, the system will have a maximum DNF compared to other cases and leads to more DC voltage to reach the pull-in instability. Also, when Lame's constants $\mu^{I,S}$ are not considered, due to decreasing of nanoshell stiffness, it has a lower DNF than the other parameters and SS MWPENS will sooner than other parameters reach the pull-in voltage.

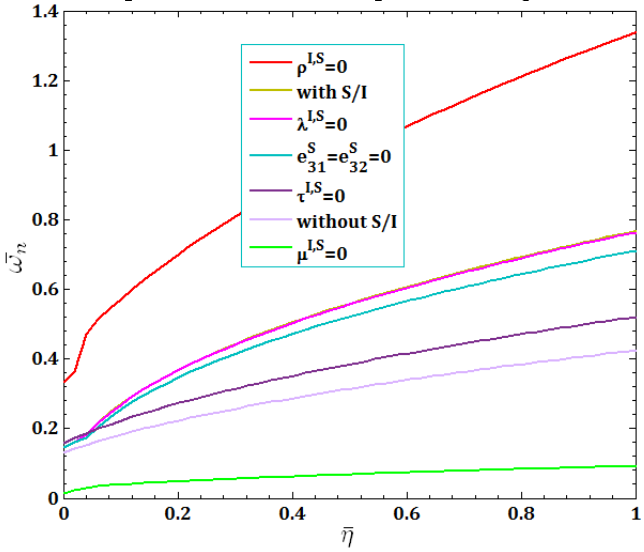

Fig. 10 The effects surface/interface parameters on DNF versus $\bar{\eta}$ with $\bar{\mu} = 0.1$ for SS MWPENS

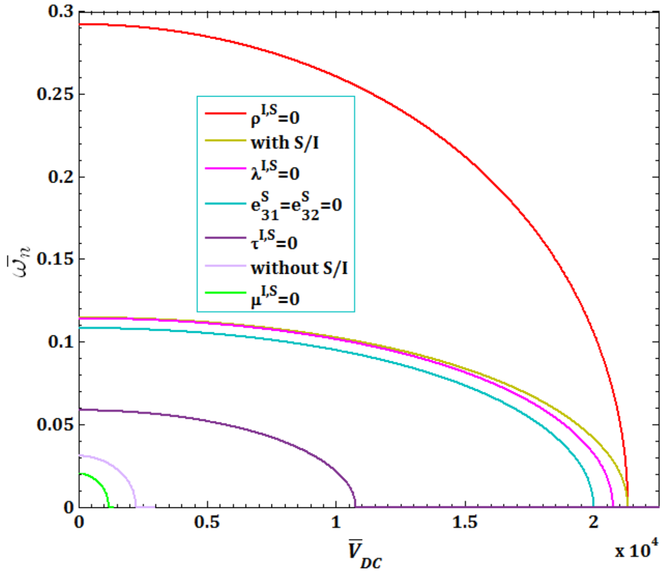

Fig. 11 The effects of surface/ interface parameters on DNF versus pull-in voltage $\bar{V}_{DC}$ with $\bar{\mu} = 0.1$ and $\bar{\eta} = 0.01$ for SS MWPENS

IV. CONCLUSION

In current study, the effect of surface/interface energy, size and material length scale parameters on pull-in voltage $\bar{V}_{DC}$ and dimensionless natural frequency are investigated using three nonclassical theories of nonlocal, nonlocal strain gradient and Gurtin–Murdoch surface/interface for vibration control of multi walled piezoelectric nanosensor. The MW piezoelectric nanosensor is embedded in nonlinear electrostatic and harmonic excitations, structural damping and nonlinear van der Waals force. Hamilton's principle and Galerkin technique respectively are used to obtain the governing equations and boundary conditions and also to solve the equation of motion. For this work, nonclassical theories compared with classical theory CT. Some conclusions are obtained from this study:

- in the case of with surface/interface effect, due to increasing of MWPENS stiffness, the dimensionless natural frequency increase compared to case of without S/I effects.
- Nonlocal scale parameter $\bar{\mu}$, material length scale parameter $\bar{\eta}$ and pull-in voltage DC respectively lead to increasing and decreasing of MWPENS stiffness and results of lead to increasing and decreasing the DNF of MWPENS.
- in all cases of nonlocal, material length scale parameters and voltage DC, the dimensionless natural frequency related to CC boundary condition is higher than that related to CS, SS and CF one.

- for all modes, due to decreasing of MWPENS stiffness, the DNF decreases when the L/R_1 ratio increases.
- with increasing of vibrational modes number, DNF increases.
- the highest frequency is related to the GMSIT and shows that in this case, the system's rigidity is higher than other cases.
- the lowest frequency is also related to nonlocal theory NL and shows that in this case, the system hardness is less than other cases.
- the natural frequency of the classical theory is greater than the NLT, NSGT and combination of GMSIT+NLT and GMSIT+NSGT theories.
- S/I effects lead to increasing of MWPENS stiffness and other nonclassical effects lead to decreasing of MWPENS stiffness and as a result decreasing of DNF.
- with increasing of piezoelectric voltage in all theories, due to increasing the rigidity of the system, the frequency increases.
- Taking into account the effects of the surface/interface causes a rigidity of the system and leads to more DC voltage to reach the pull-in voltages.
- The three theories of NLT, NSGT and CT, due to the softening of the system in these cases, respectively will soon reach the pull-in voltage.
- with ignoring the surface/interface density $\rho^{I,S}$, the inertia of the system will greatly decrease and due to increasing of SS MWPENS stiffness, the system will have a maximum DNF compared to other cases and leads to more DC voltage to reach the pull-in instability.
- when Lame's constants $\mu^{I,S}$ is not considered, due to decreasing of nanoshell stiffness, it has a lower DNF than the other parameters and SS MWPENS will sooner than other parameters reach the pull-in voltage.

Conflict of interest
The authors report no conflict of interest.

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Appendix 1

$$\alpha_{1un}=-\frac{1}{m_{3n}}\bar{A}_{11n},\alpha_{2un}=-\frac{m_{0n}^2}{m_{3n}}\bar{A}_{66n},$$

$$\alpha_{3un}=-\frac{m_{0n}}{m_{3n}}(\bar{A}_{12n}+\bar{A}_{66n}),$$

$$\alpha_{4un}=-\frac{m_{0n}}{m_{3n}}\bar{A}_{12n},$$

$$\alpha_{5un}=-\frac{1}{m_{1n}m_{3n}}\left(\bar{A}_{11n}-2\left(\bar{\tau}_{0n}^S+\bar{\tau}_{0n}^I\right)\right),$$

$$\alpha_{6un}=-\frac{m_{0n}m_{2n}}{m_{3n}}\bar{A}_{66n},$$

$$\alpha_{7un}=-\frac{m_{0n}m_{2n}}{m_{3n}}(\bar{A}_{12n}+\bar{A}_{66n}),$$

$$\alpha_{1vn}=-\frac{m_{0n}}{m_{3n}}(\bar{A}_{21n}+\bar{A}_{66n}),$$

$$\alpha_{2vn}=-\frac{1}{m_{3n}}\bar{A}_{66n},\alpha_{3vn}=-\frac{m_{0n}^2}{m_{3n}}\bar{A}_{22n},$$

$$\alpha_{4vn}=-\frac{m_{2n}}{m_{3n}}(\bar{A}_{21n}+\bar{A}_{66n}),$$

$$\alpha_{5vn}=-\frac{m_{2n}}{m_{3n}}\bar{A}_{66n},$$

$$\alpha_{6vn}=-\frac{m_{0n}^2}{m_{3n}}\left(\bar{A}_{22n}-2\left(\bar{\tau}_{0n}^S+\bar{\tau}_{0n}^I\right)\right),$$

$$\alpha_{7vn}=-\frac{m_{0n}^2m_{2n}}{m_{3n}}\left(\bar{A}_{22n}-2\left(\bar{\tau}_{0n}^S+\bar{\tau}_{0n}^I\right)\right),$$

$$\alpha_{1wn}=\frac{m_{0n}}{m_{3n}}\bar{A}_{21n},\alpha_{2wn}=-\frac{1}{m_{1n}m_{3n}}\bar{A}_{11n},$$

$$\alpha_{3wn}=-\frac{m_{0n}m_{2n}}{m_{3n}}\bar{A}_{21n},\alpha_{4wn}=-\frac{1}{m_{1n}m_{3n}}\bar{A}_{11n},$$

$$\alpha_{5wn}=-\frac{m_{0n}m_{2n}}{m_{3n}}\bar{A}_{66n},\alpha_{6wn}=-\frac{m_{0n}m_{2n}}{m_{3n}}\bar{A}_{21n},$$

$$\alpha_{7wn}=-\frac{2m_{0n}m_{2n}}{m_{3n}}\bar{A}_{66n},$$

$$\alpha_{8wn}=-\frac{m_{0n}m_{2n}}{m_{3n}}\bar{A}_{66n},\alpha_{9wn}=-\frac{2m_{2n}}{m_{3n}}\bar{A}_{66n},$$

$$\alpha_{10wn}=-\frac{m_{2n}}{m_{3n}}\bar{A}_{66n},$$

$$\alpha_{11wn}=-\frac{m_{2n}}{m_{3n}}(\bar{A}_{12n}+\bar{A}_{66n}),$$

$$\alpha_{12wn}=\frac{m_{0n}^2}{m_{3n}}\bar{A}_{22n},\alpha_{13wn}=-\frac{m_{2n}}{m_{3n}}\bar{A}_{12n},$$

$$\alpha_{14wn}=-\frac{m_{0n}^2m_{2n}}{m_{3n}}\bar{A}_{22n},$$

$$\alpha_{15wn}=-\frac{m_{0n}^2m_{2n}}{m_{3n}}\bar{A}_{22n},$$

$$\alpha_{16wn}=-\frac{m_{0n}^2}{m_{3n}}\left(2\left(\bar{\tau}_{0n}^S+\bar{\tau}_{0n}^I\right)-\bar{A}_{22n}\right),$$

$$\alpha_{17wn}=-\frac{m_{2n}}{m_{n3}}\bar{A}_{12n},$$

$$\alpha_{18wn}=-\frac{m_{0n}^2m_{2n}}{m_{3n}}\left(\bar{A}_{22n}-2\left(\bar{\tau}_{0n}^S+\bar{\tau}_{0n}^I\right)\right),$$

$$\alpha_{19wn}=-\frac{1}{m_{3n}}\left(2\left(\bar{\tau}_{0n}^S+\bar{\tau}_{0n}^I\right)-\bar{N}_{xp2}\right),$$

$$\alpha_{20wn}=-\frac{m_{2n}^2}{m_{3n}}(\bar{A}_{12n}+\bar{A}_{21n}+4\bar{A}_{66n}),$$

$$\alpha_{21wn} = -\frac{3}{2m_{1n}^2 m_{3n}}\left(\bar{A}_{11n} - 2\left(\bar{\tau}_{0n}^S + \bar{\tau}_{0n}^I\right)\right),$$
$$\alpha_{22wn} = -\frac{m_{2n}^2}{2m_{3n}}(\bar{A}_{12n} + 2\bar{A}_{66n}),$$
$$\alpha_{23wn} = -\frac{1}{m_{1n}^2 m_{3n}}(\bar{E}_{11n}^* - \bar{D}_{11n}),$$
$$\alpha_{24wn} = \frac{m_{2n}}{2m_{3n}}\bar{A}_{21n},$$
$$\alpha_{25wn} = -\frac{m_{2n}^2}{m_{3n}}(2\bar{E}_{11n}^* - \bar{D}_{12n} - 4\bar{D}_{66n}),$$
$$\alpha_{26wn} = -\frac{m_2^2 n}{2m_{3n}}(\bar{A}_{21n} + 2\bar{A}_{66n}),$$
$$\alpha_{27wn} = -\frac{m_{0n}^2}{m_{3n}}\left(2\left(\bar{\tau}_{0n}^S + \bar{\tau}_{0n}^I\right) - \bar{N}_{\theta p2}\right),$$
$$\alpha_{28wn} = -\frac{m_{0n}^2 m_{2n}^2}{m_{3n}}(\bar{E}_{11n}^* - \bar{D}_{22n}),$$
$$\alpha_{29wn} = -\frac{3m_{0n}^2 m_{2n}^2}{2m_{3n}}\left(\bar{A}_{22n} - 2\left(\bar{\tau}_{0n}^S + \bar{\tau}_{0n}^I\right)\right),$$
$$\alpha_{30wn} = -\frac{m_{0n}^2 m_{2n}}{2m_{3n}}\left(\bar{A}_{22n} - 2\left(\bar{\tau}_{0n}^S + \bar{\tau}_{0n}^I\right)\right),$$
$$\alpha_{31wn} = \frac{1}{2m_{1n}^2 m_{3n}}\bar{G}_{11n}^*, \alpha_{32wn} = \frac{m_{2n}^2}{2m_{3n}}\bar{G}_{11n}^*,$$
$$\alpha_{33wn} = -\frac{m_{0n}m_{1n}}{m_{3n}}\left(\bar{N}_{\theta p2} - 2\left(\bar{\tau}_{0n}^S + \bar{\tau}_{0n}^I\right)\right),$$

Appendix 2

$$\alpha_{1un}^{bc} = \frac{1}{m_{3n}}\bar{A}_{11n}, \alpha_{2un}^{bc} = \frac{m_{0n}}{m_{3n}}\bar{A}_{12n},$$
$$\alpha_{3un}^{bc} = \frac{m_{0n}}{m_{3n}}\bar{A}_{12n},$$
$$\alpha_{4un}^{bcn} = \frac{1}{2m_{1n}m_{3n}}\left(\bar{A}_{11n} - 2\left(\bar{\tau}_{0n}^S + \bar{\tau}_{0n}^I\right)\right),$$
$$\alpha_{5un}^{bc} = \frac{m_{0n}m_{2n}}{2m_{3n}}\bar{A}_{12n},$$
$$\alpha_{6un}^{bc} = \frac{m_{1n}}{m_{3n}}\left(2\left(\bar{\tau}_{0n}^S + \bar{\tau}_{0n}^I\right) - \bar{N}_{xp2}\right),$$
$$\alpha_{7un}^{bc} = \frac{m_0^2}{m_{3n}}\bar{A}_{66n}, \alpha_{8un}^{bc} = \frac{m_{0n}}{m_{3n}}\bar{A}_{66n},$$
$$\alpha_{9un}^{bc} = \frac{m_{0n}m_{2n}}{m_{3n}}\bar{A}_{66n}, \alpha_{1vn}^{bc} = \frac{m_0}{m_{3n}}\bar{A}_{66},$$
$$\alpha_{2vn}^{bc} = \frac{1}{m_{3n}}\bar{A}_{66n}, \alpha_{3vn}^{bc} = \frac{m_{2n}}{m_{3n}}\bar{A}_{66n},$$
$$\alpha_{4vn}^{bc} = \frac{m_{0n}}{m_{3n}}\bar{A}_{21n}, \alpha_{5vn}^{bc} = \frac{m_{0n}^2}{m_{3n}}\bar{A}_{22n},$$
$$\alpha_{6vn}^{bc} = \frac{m_{0n}^2}{m_{3n}}\left(\bar{A}_{22n} - 2\left(\bar{\tau}_{0n}^S + \bar{\tau}_{0n}^I\right)\right),$$
$$\alpha_{7vn}^{bc} = \frac{m_{2n}}{2m_{3n}}\bar{A}_{21n},$$
$$\alpha_{8vn}^{bc} = \frac{m_{0n}^2 m_{2n}}{2m_{3n}}\left(\bar{A}_{22n} - 2\left(\bar{\tau}_{0n}^S + \bar{\tau}_{0n}^I\right)\right),$$
$$\alpha_{9vn}^{bc} = \frac{m_{0n}m_{1n}}{m_{3n}}\left(2\left(\bar{\tau}_{0n}^S + \bar{\tau}_{0n}^I\right) - \bar{N}_{\theta p2}\right),$$
$$\alpha_{1wn}^{bc} = \frac{1}{m_{1n}\, m_{3n}}\bar{A}_{11n}, \alpha_{2wn}^{bc} = \frac{m_{0n}m_{2n}}{m_{3n}}\bar{A}_{66n},$$
$$\alpha_{3wn}^{bc} = \frac{m_{2n}}{m_{3n}}\bar{A}_{66n}, \alpha_{4wn}^{bc} = \frac{m_{2n}}{m_{3n}}\bar{A}_{12n},$$
$$\alpha_{5wn}^{bc} = \frac{m_{2n}}{m_{3n}}\bar{A}_{12n},$$
$$\alpha_{6wn}^{bc} = \frac{1}{m_{3n}}\left(2\left(\bar{\tau}_{0n}^S + \bar{\tau}_{0n}^I\right) - \bar{N}_{xp2}\right),$$
$$\alpha_{7wn}^{bc} = \frac{1}{m_{1n}^2 m_{3n}}(\bar{E}_{11n}^* - \bar{D}_{11n}),$$
$$\alpha_{8wn}^{bc} = \frac{m_{2n}^2}{m_{3n}}(\bar{E}_{11n}^* - \bar{D}_{12n} - 2\bar{D}_{66n}),$$
$$\alpha_{9wn}^{bc} = \frac{1}{2m_{1n}^2 m_{3n}}\left(\bar{A}_{11n} - 2\left(\bar{\tau}_{0n}^S + \bar{\tau}_{0n}^I\right)\right),$$
$$\alpha_{10wn}^{bc} = \frac{m_{2n}^2}{2m_{3n}}(\bar{A}_{12n} + 2\bar{A}_{66n}),$$
$$\alpha_{11wn}^{bc} = -\frac{1}{2m_{1n}^2 m_{3n}}\bar{G}_{11n}^*,$$
$$\alpha_{12wn}^{bc} = \frac{m_{0n}m_{2n}}{m_{3n}}\bar{A}_{21n},$$
$$\alpha_{13wn}^{bc} = \frac{m_{0n}m_{2n}}{m_{3n}}\bar{A}_{66n}, \alpha_{14wn}^{bc} = \frac{m_{2n}}{m_{3n}}\bar{A}_{66n},$$
$$\alpha_{15wn}^{bc} = \frac{m_{0n}^2\, m_{2n}}{m_{3n}}\bar{A}_{22n},$$
$$\alpha_{16wn}^{bc} = \frac{m_{0n}^2\, m_{2n}}{m_{3n}}\left(\bar{A}_{22n} - 2\left(\bar{\tau}_{0n}^S + \bar{\tau}_{0n}^I\right)\right),$$
$$\alpha_{17wn}^{bc} = \frac{m_{2n}^2}{m_{3n}}(\bar{E}_{11n}^* - \bar{D}_{21n} - 2\bar{D}_{66n}),$$
$$\alpha_{18wn}^{bc} = \frac{m_{2n}^2}{2m_{3n}}(\bar{A}_{21n} + 2\bar{A}_{66n}),$$
$$\alpha_{19wn}^{bc} = \frac{m_{0n}^2}{m_{3n}}\left(2\left(\bar{\tau}_{0n}^S + \bar{\tau}_{0n}^I\right) - \bar{N}_{\theta p2}\right),$$
$$\alpha_{20wn}^{bc} = \frac{m_{0n}^2 m_{2n}^2}{m_{3n}}(\bar{E}_{11n}^* - \bar{D}_{22n}),$$
$$\alpha_{21wn}^{bc} = \frac{m_{0n}^2 m_{2n}^2}{2m_{3n}}\left(\bar{A}_{22n} - 2\left(\bar{\tau}_{0n}^S + \bar{\tau}_{0n}^I\right)\right),$$
$$\alpha_{22wn}^{bc} = -\frac{m_{2n}^2}{2m_{3n}}\bar{G}_{11n}^*, \alpha_{23wn}^{bc} = -\frac{1}{m_{1n}^2 m_{3n}}\bar{D}_{11n},$$
$$\alpha_{24wn}^{bc} = -\frac{m_{2n}^2}{m_{3n}}\bar{D}_{12n}, \alpha_{25wn}^{bc} = -\frac{1}{m_{3n}}\bar{M}_{xp2},$$
$$\alpha_{26wn}^{bc} = \frac{1}{m_{1n}^2 m_{3n}}\bar{E}_{11n}^*, \alpha_{27wn}^{bc} = \frac{m_{2n}^2}{m_{3n}}\bar{E}_{11n}^*,$$
$$\alpha_{28wn}^{bc} = -\frac{1}{2m_{1n}^2 m_{3n}}\bar{G}_{11n}^*,$$
$$\alpha_{29wn}^{bc} = -\frac{2m_{2n}^2}{m_{3n}}\bar{D}_{66n},$$
$$\alpha_{30wn}^{bc} = -\frac{2m_{2n}^2}{m_{3n}}\bar{D}_{66n}, \alpha_{31wn}^{bc} = -\frac{m_{2n}^2}{m_{3n}}\bar{D}_{21n},$$

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$$\alpha_{32wn}^{bc} = -\frac{m_{0n}^2 m_{2n}^2}{m_{3n}} \bar{D}_{22n}, \alpha_{33wn}^{bc} = -\frac{m_{0n}^2}{m_{3n}} \bar{M}_{\theta p2},$$

$$\alpha_{34wn}^{bc} = \frac{m_{2n}^2}{m_{3n}} \bar{E}_{11n}^*, \alpha_{35wn}^{bc} = \frac{m_{0n}^2 m_{2n}^2}{m_{3n}} \bar{E}_{11n}^*,$$

$$\alpha_{36wn}^{bc} = -\frac{m_{2n}^2}{2m_{3n}} \bar{G}_{1n1}^*,$$

Appendix 3

$$(M)_{un}^u = \iint \begin{pmatrix} \chi_e \chi_i \vartheta_f \vartheta_j - \\ \bar{\mu}\left(\chi_e \chi_i'' \vartheta_f \vartheta_j + m_0^2 \chi_e \chi_i \vartheta_f \vartheta_j''\right) \end{pmatrix}_n d\xi\, d\theta,$$

$$(K)_{un}^u = \iint \begin{pmatrix} \alpha_{1u} \chi_e \chi_i'' \vartheta_f \vartheta_j + \alpha_{2u} \chi_e \chi_i \vartheta_f \vartheta_j'' \\ -\bar{\eta} \begin{pmatrix} \alpha_{1u} \begin{pmatrix} \chi_e \chi_i'''' \vartheta_f \vartheta_j + \\ m_0^2 \chi_e \chi_i'' \vartheta_f \vartheta_j'' \end{pmatrix} \\ +\alpha_{2u} \begin{pmatrix} \chi_e \chi_i'' \vartheta_f \vartheta_j'' + \\ m_0^2 \chi_e \chi_i \vartheta_f \vartheta_j'''' \end{pmatrix} \end{pmatrix} \end{pmatrix}_n d\xi\, d\theta,$$

$$(K_{bc})_{un}^u = \alpha_{1un}^{bc} (\chi_e \chi_i')\big|_0^1 \int \vartheta_f \vartheta_j d\theta$$

$$+\alpha_{7un}^{bc} \left(\vartheta_f \vartheta_j'\right)\Big|_0^{2\pi} \int \chi_e \chi_i d\xi,$$

$$(K)_{un}^v = \iint \alpha_{3un} \begin{pmatrix} \chi_e \phi_k' \vartheta_f \alpha_l' - \\ \bar{\eta}\left(\chi_e \phi_k''' \vartheta_f \alpha_l' + m_0^2 \chi_e \phi_k' \vartheta_f \alpha_l'''\right) \end{pmatrix}_n d\xi\, d\theta,$$

$$(K_{bc})_{un}^v = \alpha_{2un}^{bc} (\chi_e \phi_k)\big|_0^1 \int \vartheta_f \alpha_l' d\theta$$

$$+\alpha_{8un}^{bc} \left(\vartheta_f \alpha_l\right)\Big|_0^{2\pi} \int \chi_e \phi_k' d\xi,$$

$$(K)_{un}^w = \iint \alpha_{4un} \begin{pmatrix} \chi_e \beta_o' \vartheta_f \psi_l - \\ \bar{\eta}\left(\chi_e \beta_o''' \vartheta_f \psi_l + m_0^2 \chi_e \beta_o' \vartheta_f \psi_l''\right) \end{pmatrix}_n d\xi\, d\theta,$$

$$(K_{bc})_{un}^w = \alpha_{3u}^{bc} (\chi_e \beta_0)\big|_0^1 \int \vartheta_f \psi_l d\theta,$$

$$(NL)_{un}^w = \iint \begin{pmatrix} \alpha_{5u} \chi_e \beta_0' \beta_t'' \vartheta_f \psi_p \psi_v + \\ \alpha_{6u} \chi_e \beta_0' \beta_t \vartheta_f \psi_p \psi_v'' \\ +\alpha_{7u} \chi_e \beta_0' \beta_t \vartheta_f \psi_p' \psi_v' \\ -\bar{\eta} \begin{pmatrix} \alpha_{5u} \begin{pmatrix} \chi_e \beta_0''' \beta_t'' \vartheta_f \psi_p \psi_v + \\ \chi_e \beta_0' \beta_t'''' \vartheta_f \psi_p \psi_v \\ +m_0^2 \begin{pmatrix} \chi_e \beta_0' \beta_t'' \vartheta_f \psi_p'' \psi_v + \\ \chi_e \beta_0' \beta_t'' \vartheta_f \psi_p \psi_v'' \end{pmatrix} \end{pmatrix} \\ +\alpha_{6u} \begin{pmatrix} \chi_e \beta_0''' \beta_t \vartheta_f \psi_p \psi_v'' + \\ \chi_e \beta_0' \beta_t'' \vartheta_f \psi_p \psi_v'' \\ +m_0^2 \begin{pmatrix} \chi_e \beta_0' \beta_t \vartheta_f \psi_p'' \psi_v'' + \\ \chi_e \beta_0' \beta_t \vartheta_f \psi_p \psi_v'''' \end{pmatrix} \end{pmatrix} \\ +\alpha_{7u} \begin{pmatrix} \chi_e \beta_0''' \beta_t \vartheta_f \psi_p' \psi_v' + \\ \chi_e \beta_0' \beta_t'' \vartheta_f \psi_p' \psi_v' \\ +m_0^2 \begin{pmatrix} \chi_e \beta_0' \beta_t \vartheta_f \psi_p''' \psi_v' + \\ \chi_e \beta_0' \beta_t \vartheta_f \psi_p' \psi_v''' \end{pmatrix} \end{pmatrix} \end{pmatrix} \end{pmatrix}_n d\xi\, d\theta,$$

$$(NL_{bc})_{un}^w = \alpha_{4un}^{bc} (\chi_e \beta_0' \beta_t')\big|_0^1 \int \vartheta_f \psi_p \psi_v d\theta$$

$$+\alpha_{5un}^{bc} (\chi_e \beta_o \beta_t)\big|_0^1 \int \vartheta_f \psi_p' \psi_v' d\theta$$

$$+\alpha_{9un}^{bc} \left(\vartheta_f \psi_p \psi_v'\right)\Big|_0^{2\pi} \int \chi_e \beta_0' \beta_t d\xi,$$

$$\bar{F}_{upn}^{bc} = \alpha_{6un}^{bc} (\chi_e)\big|_0^1 \int \vartheta_f d\theta,$$

$$(M)_{vn}^v = \iint \begin{pmatrix} \phi_q \phi_k \alpha_f \alpha_l - \\ \bar{\mu}\left(\phi_q \phi_k'' \alpha_f \alpha_l + m_0^2 \phi_q \phi_k \alpha_f \alpha_l''\right) \end{pmatrix}_n d\xi\, d\theta,$$

$$(K)_{vn}^u = \iint \alpha_{1vn} \begin{pmatrix} \phi_q \chi_i' \alpha_f \vartheta_l' - \\ \bar{\eta}\left(\phi_q \chi_i''' \alpha_f \vartheta_l' + m_0^2 \phi_q \chi_i' \alpha_f \vartheta_l'''\right) \end{pmatrix}_n d\xi\, d\theta,$$

$$(K_{bc})_{vn}^u = \alpha_{1vn}^{bc} \left(\phi_q \chi_i\right)\Big|_0^1 \int \alpha_f \vartheta_l' d\theta +$$

$$\alpha_{4vn}^{bc} \left(\alpha_f \vartheta_l\right)\Big|_0^{2\pi} \int \phi_q \chi_i' d\xi,$$

$$(K)_{vn}^v = \iint \begin{pmatrix} \alpha_{2v} \phi_q \phi_k'' \alpha_f \alpha_l + \alpha_{3v} \phi_q \phi_k \alpha_f \alpha_l'' \\ -\bar{\eta} \begin{pmatrix} \alpha_{2v} \left(\phi_q \phi_k'''' \alpha_f \alpha_l + m_0^2 \phi_q \phi_k'' \alpha_f \alpha_l''\right) \\ +\alpha_{3v} \left(\phi_q \phi_k'' \alpha_f \alpha_l'' + m_0^2 \phi_q \phi_k \alpha_f \alpha_l''''\right) \end{pmatrix} \end{pmatrix}_n d\xi\, d\theta,$$

$$(K_{bc})_{vn}^v = \alpha_{2vn}^{bc} \left(\phi_q \phi_k'\right)\Big|_0^1 \int \alpha_f \alpha_l d\theta +$$

$$\alpha_{5vn}^{bc}\left(\alpha_f\alpha_l'\right)\Big|_0^{2\pi}\int\phi_q\phi_k d\xi,$$

$$(K)_{vn}^{w}=$$
$$\alpha_{6vn}\iint\begin{pmatrix}\phi_q\beta_o\alpha_f\psi_l'-\\ \bar{\eta}\left(\phi_q\beta_o''\alpha_f\psi_l'+m_0^2\phi_q\beta_o\alpha_f\psi_l'''\right)\end{pmatrix}_n d\xi\, d\theta,$$

$$(K_{bc})_{vn}^{w}=\alpha_{6vn}^{bc}\left(\alpha_f\psi_l\right)\Big|_0^{2\pi}\int\phi_q\beta_o d\xi,$$

$$(NL)_{vn}^{w}=$$
$$\iint\begin{pmatrix}\alpha_{4v}\phi_q\beta_o'\beta_t'\alpha_f\psi_p\psi_v'+\\ \alpha_{5v}\phi_q\beta_0''\beta_t\alpha_f\psi_p\psi_v'\\ +\alpha_{7v}\phi_q\beta_o\beta_t\alpha_f\psi_p'\psi_v''\\ -\bar{\eta}\begin{pmatrix}\alpha_{4v}\times\\ \begin{pmatrix}\phi_q\beta_o'''\beta_t'\alpha_f\psi_p\psi_v'+\\ \phi_q\beta_o'\beta_t'''\alpha_f\psi_p\psi_v'\\ +m_0^2\begin{pmatrix}\phi_q\beta_o'\beta_t'\alpha_f\psi_p''\psi_v'+\\ \phi_q\beta_o'\beta_t'\alpha_f\psi_p\psi_v'''\end{pmatrix}\end{pmatrix}\\ +\alpha_{5v}\times\\ \begin{pmatrix}\phi_q\beta_0''''\beta_t\alpha_f\psi_p\psi_v'+\\ \phi_q\beta_0''\beta_t''\alpha_f\psi_p\psi_v'\\ +m_0^2\begin{pmatrix}\phi_q\beta_0''\beta_t\alpha_f\psi_p''\psi_v'+\\ \phi_q\beta_0''\beta_t\alpha_f\psi_p\psi_v'''\end{pmatrix}\end{pmatrix}\\ +\alpha_{7v}\times\\ \begin{pmatrix}\phi_q\beta_o''\beta_t\alpha_f\psi_p'\psi_v''+\\ \phi_q\beta_o\beta_t''\alpha_f\psi_p'\psi_v''\\ +m_0^2\begin{pmatrix}\phi_q\beta_o\beta_t\alpha_f\psi_p'''\psi_v''+\\ \phi_q\beta_o\beta_t\alpha_f\psi_p'\psi_v''''\end{pmatrix}\end{pmatrix}\end{pmatrix}\end{pmatrix}_n d\xi\, d\theta,$$

$$(NL_{bc})_{vn}^{w}=\alpha_{3vn}^{bc}\left(\phi_q\beta_o'\beta_t\right)\Big|_0^{1}\int\alpha_f\psi_p\psi_v' d\theta+$$

$$\alpha_{7vn}^{bc}\left(\alpha_f\psi_p\psi_v\right)\Big|_0^{2\pi}\int\phi_q\beta_o'\beta_t' d\xi$$

$$+\alpha_{8vn}^{bc}\left(\alpha_f\psi_p'\psi_v'\right)\Big|_0^{2\pi}\int\phi_q\beta_o\beta_t d\xi,$$

$$\bar{F}_{vpn}^{bc}=\alpha_{9vn}^{bc}\left(\alpha_f\right)\Big|_0^{2\pi}\int\phi_q d\xi,$$

$$(M)_{wn}^{w}=$$
$$\iint\begin{pmatrix}\beta_r\beta_o\psi_s\psi_p+\alpha_{31w}\beta_r\beta_o''\psi_s\psi_p+\\ \alpha_{32w}\beta_r\beta_o\psi_s\psi_p''\\ -\bar{\mu}\left(\beta_r\beta_o''\psi_s\psi_p+m_0^2\beta_r\beta_o\psi_s\psi_p''\right)\end{pmatrix}_n d\xi\, d\theta,$$

$$(M_{bc})_{wn}^{w}=\alpha_{11wn}^{bc}(\beta_r\beta_o')|_0^1\int\psi_s\psi_p d\theta+$$

$$\alpha_{28wn}^{bc}(\beta_r\beta_o)|_0^1\int\psi_s\psi_p d\theta$$

$$+\alpha_{36wn}^{bc}\left(\psi_s\psi_p\right)\Big|_0^{2\pi}\int\beta_r\beta_o d\xi,$$

$$(C)_{wn}^{w}=$$
$$\iint\bar{C}_{wn}\begin{pmatrix}\beta_r\beta_o\psi_s\psi_p\\ -\bar{\mu}\left(\beta_r\beta_o''\psi_s\psi_p+m_0^2\beta_r\beta_o\psi_s\psi_p''\right)\end{pmatrix}_n d\xi\, d\theta,$$

$$(K)_{wn}^{u}=$$
$$\iint\alpha_{1wn}\begin{pmatrix}\beta_r\chi_i'\psi_s\vartheta_j-\\ \bar{\eta}\left(\beta_r\chi_i'''\psi_s\vartheta_j+m_0^2\beta_r\chi_i'\psi_s\vartheta_j''\right)\end{pmatrix}_n d\xi\, d\theta,$$

$$(K)_{wn}^{v}=$$
$$\iint\alpha_{12wn}\begin{pmatrix}\beta_r\phi_k\psi_s\alpha_l'-\\ \bar{\eta}\left(\beta_r\phi_k''\psi_s\alpha_l'+m_0^2\beta_r\phi_k\psi_s\alpha_l'''\right)\end{pmatrix}_n d\xi\, d\theta,$$

$$(K)_{wn}^{w}=$$
$$\iint\begin{pmatrix}\alpha_{16w}\beta_r\beta_o\psi_s\psi_p+\alpha_{19w}\beta_r\beta_0''\psi_s\psi_p+\\ \alpha_{23w}\beta_r\beta_o''''\psi_s\psi_p\\ +\alpha_{25w}\beta_r\beta_o''\psi_s\psi_p''+\alpha_{27w}\beta_r\beta_o\psi_s\psi_p''+\\ \alpha_{28w}\beta_r\beta_o\psi_s\psi_p''''\\ -\bar{\eta}\times\\ \begin{pmatrix}\alpha_{16w}\left(\beta_r\beta_o''\psi_s\psi_p+m_0^2\beta_r\beta_o\psi_s\psi_p''\right)\\ +\alpha_{19w}\left(\beta_r\beta_0''''\psi_s\psi_p+m_0^2\beta_r\beta_0''\psi_s\psi_p''\right)\\ +\alpha_{23w}\left(\beta_r\beta_o''''''\psi_s\psi_p+m_0^2\beta_r\beta_o''''\psi_s\psi_p''\right)\\ +\alpha_{25w}\left(\beta_r\beta_o''''\psi_s\psi_p''+m_0^2\beta_r\beta_o''\psi_s\psi_p''''\right)\\ +\alpha_{27w}\left(\beta_r\beta_o''\psi_s\psi_p''+m_0^2\beta_r\beta_o\psi_s\psi_p''''\right)\\ +\alpha_{28w}\left(\beta_r\beta_o''\psi_s\psi_p''''+m_0^2\beta_r\beta_o\psi_s\psi_p''''''\right)\end{pmatrix}\end{pmatrix}_n d\xi\, d\theta,$$

$$(K_{bc})_{wn}^{w}=\alpha_{6wn}^{bc}(\beta_r\beta_o')|_0^1\int\psi_s\psi_p d\theta+$$

$$\alpha_{7wn}^{bc}(\beta_r\beta_o''')|_0^1\int\psi_s\psi_p d\theta+$$

$$\alpha_{8wn}^{bc}(\beta_r\beta_o')|_0^1\int\psi_s\psi_p'' d\theta$$

$$+\alpha_{17wn}^{bc}\left(\psi_s\psi_p'\right)\Big|_0^{2\pi}\int\beta_r\beta_o'' d\xi+$$

$$\alpha_{19wn}^{bc}\left(\psi_s\psi_p'\right)\Big|_0^{2\pi}\int\beta_r\beta_o d\xi+$$

$$\alpha_{20wn}^{bc}\left(\psi_s\psi_p'''\right)\Big|_0^{2\pi}\int\beta_r\beta_o d\xi$$

$$+\alpha_{23wn}^{bc}(\beta_r\beta_o'')|_0^1\int\psi_s\psi_p d\theta$$

$$+\alpha_{24wn}^{bc}(\beta_r\beta_o)|_0^1 \int \psi_s\psi_p''d\theta$$

$$+\alpha_{26wn}^{bc}(\beta_r\beta_o'')|_0^1 \int \psi_s\psi_p d\theta$$

$$+\alpha_{27wn}^{bc}(\beta_r\beta_o)|_0^1 \int \psi_s\psi_p''d\theta$$

$$+\alpha_{29wn}^{bc}\left(\psi_s\psi_p'\right)\Big|_0^{2\pi} \int \beta_r\beta_o'd\xi$$

$$+\alpha_{30wn}^{bc}(\beta_r\beta_o')|_0^1 \int \psi_s\psi_p'd\theta$$

$$+\alpha_{31wn}^{bc}\left(\psi_s\psi_p\right)\Big|_0^{2\pi} \int \beta_r\beta_o''d\xi$$

$$+\alpha_{32wn}^{bc}\left(\psi_s\psi_p''\right)\Big|_0^{2\pi} \int \beta_r\beta_o d\xi$$

$$+\alpha_{34wn}^{bc}\left(\psi_s\psi_p\right)\Big|_0^{2\pi} \int \beta_r\beta_o''d\xi$$

$$+\alpha_{35wn}^{bc}\left(\psi_s\psi_p''\right)\Big|_0^{2\pi} \int \beta_r\beta_o d\xi,$$

$$(K_e)_{wn}^{w} = \iint \left(\begin{array}{c}\beta_r\beta_o\psi_s\psi_p - \\ \bar{\mu}\left(\beta_r\beta_o''\psi_s\psi_p + m_0^2\beta_r\beta_o\psi_s\psi_p''\right)\end{array}\right)_n d\xi\, d\theta,$$

$$(K_{e2})_{wn}^{w} = -\bar{C}_2\bar{F}_e\bar{V}_{DC}^2(K_e)_{wn}^{w},$$

$$(NL)_{wn}^{u} =$$

$$\iint \left(\begin{array}{c}
\alpha_{2w}\beta_r\beta_o''\chi_i'\psi_s\psi_p\vartheta_j + \alpha_{3w}\beta_r\beta_o\chi_i'\psi_s\psi_p''\vartheta_j + \\
\alpha_{4w}\beta_r\beta_o'\chi_i''\psi_s\psi_p\vartheta_j \\
+\alpha_{5w}\beta_r\beta_o\chi_i'\psi_s\psi_p'\vartheta_j' + \alpha_{6w}\beta_r\beta_o\chi_i'\psi_s\psi_p'\vartheta_j' + \\
\alpha_{7w}\beta_r\beta_o'\chi_i\psi_s\psi_p'\vartheta_j' \\
+\alpha_{8w}\beta_r\beta_o'\chi_i\psi_s\psi_p\vartheta_j'' \\
-\bar{\eta}\times \\
\left(\begin{array}{c}
\alpha_{2w}\left(\begin{array}{c}\beta_r\beta_o''''\chi_i'\psi_s\psi_p\vartheta_j + \\ \beta_r\beta_o''\chi_i'''\psi_s\psi_p\vartheta_j \\ +m_0^2\left(\begin{array}{c}\beta_r\beta_o''\chi_i'\psi_s\psi_p''\vartheta_j + \\ \beta_r\beta_o''\chi_i'\psi_s\psi_p\vartheta_j''\end{array}\right)\end{array}\right) \\
+\alpha_{3w}\left(\begin{array}{c}\beta_r\beta_o''\chi_i'\psi_s\psi_p''\vartheta_j + \\ \beta_r\beta_o\chi_i'''\psi_s\psi_p''\vartheta_j \\ +m_0^2\left(\begin{array}{c}\beta_r\beta_o\chi_i'\psi_s\psi_p''''\vartheta_j + \\ \beta_r\beta_o\chi_i'\psi_s\psi_p''\vartheta_j''\end{array}\right)\end{array}\right) \\
+\alpha_{4w}\left(\begin{array}{c}\beta_r\beta_o'''\chi_i''\psi_s\psi_p\vartheta_j + \\ \beta_r\beta_o'\chi_i''''\psi_s\psi_p\vartheta_j \\ +m_0^2\left(\begin{array}{c}\beta_r\beta_o'\chi_i''\psi_s\psi_p''\vartheta_j + \\ \beta_r\beta_o'\chi_i''\psi_s\psi_p\vartheta_j''\end{array}\right)\end{array}\right) \\
+\alpha_{5w}\left(\begin{array}{c}\beta_r\beta_o''\chi_i'\psi_s\psi_p'\vartheta_j' + \\ \beta_r\beta_o\chi_i'''\psi_s\psi_p'\vartheta_j' \\ +m_0^2\left(\begin{array}{c}\beta_r\beta_o\chi_i'\psi_s\psi_p'''\vartheta_j' + \\ \beta_r\beta_o\chi_i'\psi_s\psi_p'\vartheta_j'''\end{array}\right)\end{array}\right) \\
+\alpha_{6w}\left(\begin{array}{c}\beta_r\beta_o''\chi_i'\psi_s\psi_p'\vartheta_j' + \\ \beta_r\beta_o\chi_i'''\psi_s\psi_p'\vartheta_j' \\ +m_0^2\left(\begin{array}{c}\beta_r\beta_o\chi_i'\psi_s\psi_p'''\vartheta_j' + \\ \beta_r\beta_o\chi_i'\psi_s\psi_p'\vartheta_j'''\end{array}\right)\end{array}\right) \\
+\alpha_{7w}\left(\begin{array}{c}\beta_r\beta_o'''\chi_i\psi_s\psi_p'\vartheta_j' + \\ \beta_r\beta_o'\chi_i''\psi_s\psi_p'\vartheta_j' \\ +m_0^2\left(\begin{array}{c}\beta_r\beta_o'\chi_i\psi_s\psi_p'''\vartheta_j' + \\ \beta_r\beta_o'\chi_i\psi_s\psi_p'\vartheta_j'''\end{array}\right)\end{array}\right) \\
+\alpha_{8w}\left(\begin{array}{c}\beta_r\beta_o'''\chi_i\psi_s\psi_p\vartheta_j'' + \\ \beta_r\beta_o'\chi_i''\psi_s\psi_p\vartheta_j'' \\ +m_0^2\left(\begin{array}{c}\beta_r\beta_o'\chi_i\psi_s\psi_p''\vartheta_j'' + \\ \beta_r\beta_o'\chi_i\psi_s\psi_p\vartheta_j''''\end{array}\right)\end{array}\right)
\end{array}\right)
\end{array}\right)_n d\xi\, d\theta,$$

$$(NL_{bc})_{wn}^{u} = \alpha_{1wn}^{bc}(\beta_r\beta_o'\chi_i')|_0^1 \int \psi_s\psi_p\vartheta_j d\theta +$$

$$\alpha_{2wn}^{bc}(\beta_r\beta_o\chi_i)|_0^1 \int \psi_s\psi_p'\vartheta_j'd\theta$$

$$+\alpha_{12wn}^{bc}\left(\psi_s\psi_p'\vartheta_j\right)\Big|_0^{2\pi} \int \beta_r\beta_o\chi_i'd\xi$$

$$+\alpha^{bc}_{13wn}\left(\psi_s\psi_p\vartheta'_j\right)\Big|_0^{2\pi}\int \beta_r\beta'_o\chi_i d\xi,$$

$$(NL)^v_{wn}=$$

$$\iint\left(\begin{matrix}\alpha_{9w}\beta_r\beta'_o\phi'_k\psi_s\psi'_p\alpha_l+\\ \alpha_{10w}\beta_r\beta_o\phi''_k\psi_s\psi'_p\alpha_l\\ +\alpha_{11w}\beta_r\beta'_o\phi'_k\psi_s\psi_p\alpha'_l+\\ \alpha_{13w}\beta_r\beta''_o\phi_k\psi_s\psi_p\alpha'_l\\ +\alpha_{14w}\beta_r\beta_o\phi_k\psi_s\psi''_p\alpha'_l+\\ \alpha_{15w}\beta_r\beta_o\phi_k\psi_s\psi'_p\alpha''_l\\ -\bar{\eta}\times\\ \left(\begin{matrix}\alpha_{9w}\left(\begin{matrix}\beta_r\beta'''_o\phi'_k\psi_s\psi'_p\alpha_l+\\ \beta_r\beta'_o\phi'''_k\psi_s\psi'_p\alpha_l\\ +m_0^2\left(\begin{matrix}\beta_r\beta'_o\phi'_k\psi_s\psi'''_p\alpha_l+\\ \beta_r\beta'_o\phi'_k\psi_s\psi'_p\alpha''_l\end{matrix}\right)\end{matrix}\right)\\ +\alpha_{10w}\left(\begin{matrix}\beta_r\beta''_o\phi''_k\psi_s\psi'_p\alpha_l+\\ \beta_r\beta_o\phi''''_k\psi_s\psi'_p\alpha_l\\ +m_0^2\left(\begin{matrix}\beta_r\beta_o\phi''_k\psi_s\psi'''_p\alpha_l+\\ \beta_r\beta_o\phi''_k\psi_s\psi'_p\alpha''_l\end{matrix}\right)\end{matrix}\right)\\ +\alpha_{11w}\left(\begin{matrix}\beta_r\beta'''_o\phi'_k\psi_s\psi_p\alpha'_l+\\ \beta_r\beta'_o\phi'''_k\psi_s\psi_p\alpha'_l\\ +m_0^2\left(\begin{matrix}\beta_r\beta'_o\phi'_k\psi_s\psi''_p\alpha'_l+\\ \beta_r\beta'_o\phi'_k\psi_s\psi_p\alpha'''_l\end{matrix}\right)\end{matrix}\right)\\ +\alpha_{13w}\left(\begin{matrix}\beta_r\beta''''_o\phi_k\psi_s\psi_p\alpha'_l+\\ \beta_r\beta''_o\phi''_k\psi_s\psi_p\alpha'_l\\ +m_0^2\left(\begin{matrix}\beta_r\beta''_o\phi_k\psi_s\psi''_p\alpha'_l+\\ \beta_r\beta''_o\phi_k\psi_s\psi_p\alpha'''_l\end{matrix}\right)\end{matrix}\right)\\ +\alpha_{14w}\left(\begin{matrix}\beta_r\beta''_o\phi_k\psi_s\psi''_p\alpha'_l+\\ \beta_r\beta_o\phi''_k\psi_s\psi''_p\alpha'_l\\ +m_0^2\left(\begin{matrix}\beta_r\beta_o\phi_k\psi_s\psi''''_p\alpha'_l+\\ \beta_r\beta_o\phi_k\psi_s\psi''_p\alpha'''_l\end{matrix}\right)\end{matrix}\right)\\ +\alpha_{15w}\left(\begin{matrix}\beta_r\beta''_o\phi_k\psi_s\psi'_p\alpha''_l+\\ \beta_r\beta_o\phi''_k\psi_s\psi'_p\alpha''_l\\ +m_0^2\left(\begin{matrix}\beta_r\beta_o\phi_k\psi_s\psi'''_p\alpha''_l+\\ \beta_r\beta_o\phi_k\psi_s\psi'_p\alpha''''_l\end{matrix}\right)\end{matrix}\right)\end{matrix}\right)\end{matrix}\right)_n d\xi\, d\theta$$

$$(NL_{bc})^v_{wn}=\alpha^{bc}_{3wn}(\beta_r\beta_o\phi'_k)|_0^1\int \psi_s\psi'_p\alpha_l d\theta$$

$$+\alpha^{bc}_{4wn}(\beta_r\beta'_o\phi_k)|_0^1\int \psi_s\psi_p\alpha'_l d\theta$$

$$+\alpha^{bc}_{14wn}\left(\psi_s\psi_p\alpha_l\right)\Big|_0^{2\pi}\int \beta_r\beta'_o\phi'_k d\xi$$

$$+\alpha^{bc}_{15wn}\left(\psi_s\psi'_p\alpha'_l\right)\Big|_0^{2\pi}\int \beta_r\beta_o\phi_k d\xi,$$

$$(NL)^w_{w2n}=$$

$$\iint\left(\begin{matrix}\alpha_{17w}\beta_r\beta_o\beta''_t\psi_s\psi_p\psi_v+\alpha_{18w}\beta_r\beta_o\beta_t\psi_s\psi_p\psi''_v\\ +\alpha_{24w}\beta_r\beta'_o\beta'_t\psi_s\psi_p\psi_v+\alpha_{30w}\beta_r\beta_o\beta_t\psi_s\psi'_p\psi'_v\\ -\bar{\eta}\times\\ \left(\begin{matrix}\alpha_{17w}\left(\begin{matrix}\beta_r\beta''_o\beta''_t\psi_s\psi_p\psi_v+\\ \beta_r\beta_o\beta''''_t\psi_s\psi_p\psi_v\\ +m_0^2\left(\begin{matrix}\beta_r\beta_o\beta''_t\psi_s\psi''_p\psi_v+\\ \beta_r\beta_o\beta''_t\psi_s\psi_p\psi''_v\end{matrix}\right)\end{matrix}\right)\\ +\alpha_{18w}\left(\begin{matrix}\beta_r\beta''_o\beta_t\psi_s\psi_p\psi''_v+\\ \beta_r\beta_o\beta''_t\psi_s\psi_p\psi''_v\\ +m_0^2\left(\begin{matrix}\beta_r\beta_o\beta_t\psi_s\psi''_p\psi''_v+\\ \beta_r\beta_o\beta_t\psi_s\psi_p\psi''''_v\end{matrix}\right)\end{matrix}\right)\\ +\alpha_{24w}\left(\begin{matrix}\beta_r\beta'''_o\beta'_t\psi_s\psi_p\psi_v+\\ \beta_r\beta'_o\beta'''_t\psi_s\psi_p\psi_v\\ +m_0^2\left(\begin{matrix}\beta_r\beta'_o\beta'_t\psi_s\psi''_p\psi_v+\\ \beta_r\beta'_o\beta'_t\psi_s\psi_p\psi''_v\end{matrix}\right)\end{matrix}\right)\\ +\alpha_{30w}\left(\begin{matrix}\beta_r\beta''_o\beta_t\psi_s\psi'_p\psi'_v+\\ \beta_r\beta_o\beta''_t\psi_s\psi'_p\psi'_v\\ +m_0^2\left(\begin{matrix}\beta_r\beta_o\beta_t\psi_s\psi'''_p\psi'_v+\\ \beta_r\beta_o\beta_t\psi_s\psi'_p\psi'''_v\end{matrix}\right)\end{matrix}\right)\end{matrix}\right)\end{matrix}\right) d\xi\, d\theta,$$

$$(NL_{bc})^w_{w2n}=\alpha^{bc}_{5wn}(\beta_r\beta_o\beta'_t)|_0^1\int \psi_s\psi_p\psi_v d\theta$$

$$+\alpha^{bc}_{16wn}\left(\psi_s\psi_p\psi'_v\right)\Big|_0^{2\pi}\int \beta_r\beta_o\beta_t d\xi,$$

$$(NL_e)^w_{w2n}=$$

$$\iint\left(\begin{matrix}\beta_r\beta_o\beta_t\psi_s\psi_p\psi_v\\ -\bar{\mu}\left(\begin{matrix}\beta_r\beta''_o\beta_t\psi_s\psi_p\psi_v+\\ \beta_r\beta_o\beta''_t\psi_s\psi_p\psi_v\\ m_0^2\left(\begin{matrix}\beta_r\beta_o\beta_t\psi_s\psi''_p\psi_v+\\ \beta_r\beta_o\beta_t\psi_s\psi_p\psi''_v\end{matrix}\right)\end{matrix}\right)\end{matrix}\right)_n d\xi\, d\theta\,,$$

$$(NL_{2e})^w_{w2n}=-\bar{C}_3\bar{F}_e\bar{V}^2_{DC}(NL_e)^w_{w2n},$$

$$(NL)^w_{w3n}=$$

$$\iint \begin{pmatrix} \alpha_{20w}\beta_r\beta_o'\beta_t'\beta_a\psi_s\psi_p\psi_v'\psi_b' + \\ \alpha_{21w}\beta_r\beta_o''\beta_t'\beta_a'\psi_s\psi_p\psi_v\psi_b \\ +\alpha_{22w}\beta_r\beta_o''\beta_t\beta_a\psi_s\psi_p\psi_v'\psi_b' + \\ \alpha_{26w}\beta_r\beta_o'\beta_t'\beta_a\psi_s\psi_p\psi_v\psi_b'' \\ +\alpha_{29w}\beta_r\beta_o\beta_t\beta_a\psi_s\psi_p''\psi_v'\psi_b' \\ -\bar{\eta}\times \\ \begin{pmatrix} \alpha_{20w}\times \\ \begin{pmatrix} \beta_r\beta_o'''\beta_t'\beta_a\psi_s\psi_p\psi_v'\psi_b' + \\ \beta_r\beta_o'\beta_t'''\beta_a\psi_s\psi_p\psi_v'\psi_b' \\ +\beta_r\beta_o'\beta_t'\beta_a''\psi_s\psi_p\psi_v'\psi_b' \\ +m_0^2\begin{pmatrix} \beta_r\beta_o'\beta_t'\beta_a\psi_s\psi_p''\psi_v'\psi_b' + \\ \beta_r\beta_o'\beta_t'\beta_a\psi_s\psi_p\psi_v'''\psi_b' \\ +\beta_r\beta_o'\beta_t'\beta_a\psi_s\psi_p\psi_v'\psi_b''' \end{pmatrix} \end{pmatrix} \\ +\alpha_{21w}\times \\ \begin{pmatrix} \beta_r\beta_o''''\beta_t'\beta_a'\psi_s\psi_p\psi_v\psi_b + \\ \beta_r\beta_o''\beta_t'''\beta_a'\psi_s\psi_p\psi_v\psi_b \\ +\beta_r\beta_o''\beta_t'\beta_a'''\psi_s\psi_p\psi_v\psi_b \\ +m_0^2\begin{pmatrix} \beta_r\beta_o''\beta_t'\beta_a'\psi_s\psi_p''\psi_v\psi_b + \\ \beta_r\beta_o''\beta_t'\beta_a'\psi_s\psi_p\psi_v''\psi_b \\ +\beta_r\beta_o''\beta_t'\beta_a'\psi_s\psi_p\psi_v\psi_b'' \end{pmatrix} \end{pmatrix} \\ +\alpha_{22w}\times \\ \begin{pmatrix} \beta_r\beta_o''''\beta_t\beta_a\psi_s\psi_p\psi_v'\psi_b' + \\ \beta_r\beta_o''\beta_t''\beta_a\psi_s\psi_p\psi_v'\psi_b' \\ +\beta_r\beta_o''\beta_t\beta_a''\psi_s\psi_p\psi_v'\psi_b' \\ +m_0^2\begin{pmatrix} \beta_r\beta_o''\beta_t\beta_a\psi_s\psi_p''\psi_v'\psi_b' + \\ \beta_r\beta_o''\beta_t\beta_a\psi_s\psi_p\psi_v'''\psi_b' \\ +\beta_r\beta_o''\beta_t\beta_a\psi_s\psi_p\psi_v'\psi_b''' \end{pmatrix} \end{pmatrix} \\ +\alpha_{26w}\times \\ \begin{pmatrix} \beta_r\beta_o'''\beta_t'\beta_a\psi_s\psi_p\psi_v\psi_b'' + \\ \beta_r\beta_o'\beta_t'''\beta_a\psi_s\psi_p\psi_v\psi_b'' \\ +\beta_r\beta_o'\beta_t'\beta_a''\psi_s\psi_p\psi_v\psi_b'' \\ +m_0^2\begin{pmatrix} \beta_r\beta_o'\beta_t'\beta_a\psi_s\psi_p''\psi_v\psi_b'' + \\ \beta_r\beta_o'\beta_t'\beta_a\psi_s\psi_p\psi_v''\psi_b'' \\ +\beta_r\beta_o'\beta_t'\beta_a\psi_s\psi_p\psi_v\psi_b'''' \end{pmatrix} \end{pmatrix} \\ +\alpha_{29w}\times \\ \begin{pmatrix} \beta_r\beta_o''\beta_t\beta_a\psi_s\psi_p''\psi_v'\psi_b' + \\ \beta_r\beta_o\beta_t''\beta_a\psi_s\psi_p''\psi_v'\psi_b' \\ +\beta_r\beta_o\beta_t\beta_a''\psi_s\psi_p''\psi_v'\psi_b' \\ +m_0^2\begin{pmatrix} \beta_r\beta_o\beta_t\beta_a\psi_s\psi_p''''\psi_v'\psi_b' + \\ \beta_r\beta_o\beta_t\beta_a\psi_s\psi_p''\psi_v'''\psi_b' \\ +\beta_r\beta_o\beta_t\beta_a\psi_s\psi_p''\psi_v'\psi_b''' \end{pmatrix} \end{pmatrix} \end{pmatrix} \end{pmatrix}_n d\xi\, d\theta,$$

$$(NL_{bc})_{w3n}^{w} = \alpha_{9wn}^{bc}(\beta_r\beta_o'\beta_t'\beta_a')|_0^1 \int \psi_s\psi_p\psi_v\psi_b d\theta$$

$$+\alpha_{10wn}^{bc}(\beta_r\beta_o'\beta_t\beta_a)|_0^1 \int \psi_s\psi_p\psi_v'\psi_b' d\theta$$

$$+\alpha_{18wn}^{bc}\left(\psi_s\psi_p\psi_v\psi_b'\right)\Big|_0^{2\pi} \int \beta_r\beta_o'\beta_t'\beta_a d\xi$$

$$+\alpha_{21wn}^{bc}\left(\psi_s\psi_p'\psi_v'\psi_b'\right)\Big|_0^{2\pi} \int \beta_r\beta_o\beta_t\beta_a d\xi$$

$$(NL_3)_{w3n}^{w} = \iint \begin{pmatrix} \beta_r\beta_o\beta_t\beta_a\psi_s\psi_p\psi_v\psi_b \\ -\bar{\mu}\begin{pmatrix} \beta_r\beta_o''\beta_t\beta_a\psi_s\psi_p\psi_v\psi_b + \\ \beta_r\beta_o\beta_t''\beta_a\psi_s\psi_p\psi_v\psi_b \\ +\beta_r\beta_o\beta_t\beta_a''\psi_s\psi_p\psi_v\psi_b \\ +m_0^2\begin{pmatrix} \beta_r\beta_o\beta_t\beta_a\psi_s\,\psi_p''\psi_v\psi_b + \\ \beta_r\beta_o\beta_t\beta_a\psi_s\psi_p\psi_v''\psi_b \\ +\beta_r\beta_o\beta_t\beta_a\psi_s\psi_p\psi_v\psi_b'' \end{pmatrix} \end{pmatrix} \end{pmatrix}_n d\xi\, d\theta,$$

$$(NL_{3e})_{w3n}^{w} = -\bar{C}_4\bar{F}_e\bar{V}_{DC}^2(NL_{3n})_{w3}^{w},$$

$$(NL)_{w1}^{vdw} = \frac{1}{2}\iint \begin{pmatrix} \beta_{2r}\beta_{2o}\beta_{2t}\beta_{an}\psi_{2s}\psi_{2p}\psi_{2v}\psi_{bn} \\ -\mu^2\begin{pmatrix} \beta_{2r}\beta_{2o}\beta_{2t}\beta_{an}''\psi_{2s}\psi_{2p}\psi_{2v}\psi_{bn} \\ +m_0^2\beta_{2r}\beta_{2o}\beta_{2t}\beta_{an}\psi_{2s}\psi_{2p}\psi_{2v}\psi_{bn}'' \end{pmatrix} \end{pmatrix} d\xi\, d\theta,$$

$$(NL)_{w2}^{vdw} = \frac{1}{2}\iint \begin{pmatrix} \beta_{2r}\beta_{2o}\beta_{1t}\beta_{an}\psi_{2s}\psi_{2p}\psi_{1v}\psi_{bn} \\ -\mu^2\begin{pmatrix} \beta_{2r}\beta_{2o}\beta_{2t}\beta_{an}''\psi_{2s}\psi_{2p}\psi_{2v}\psi_{bn} \\ +m_0^2\beta_{2r}\beta_{2o}\beta_{2t}\beta_{an}\psi_{2s}\psi_{2p}\psi_{2v}\psi_{bn}'' \end{pmatrix} \end{pmatrix} d\xi\, d\theta,$$

$$(NL)_{w3}^{vdw} = \frac{1}{2}\iint \begin{pmatrix} \beta_{2r}\beta_{1o}\beta_{1t}\beta_{an}\psi_{2s}\psi_{1p}\psi_{1v}\psi_{bn} \\ -\mu^2\begin{pmatrix} \beta_{2r}\beta_{2o}\beta_{2t}\beta_{an}''\psi_{2s}\psi_{2p}\psi_{2v}\psi_{bn} \\ +m_0^2\beta_{2r}\beta_{2o}\beta_{2t}\beta_{an}\psi_{2s}\psi_{2p}\psi_{2v}\psi_{bn}'' \end{pmatrix} \end{pmatrix} d\xi\, d\theta,$$

$$(NL)_{w4}^{vdw} = \frac{1}{2}\iint \begin{pmatrix} \beta_{1r}\beta_{1o}\beta_{1t}\beta_{an}\psi_{1s}\psi_{1p}\psi_{1v}\psi_{bn} \\ -\mu^2\begin{pmatrix} \beta_{2r}\beta_{2o}\beta_{2t}\beta_{an}''\psi_{2s}\psi_{2p}\psi_{2v}\psi_{bn} \\ +m_0^2\beta_{2r}\beta_{2o}\beta_{2t}\beta_{an}\psi_{2s}\psi_{2p}\psi_{2v}\psi_{bn}'' \end{pmatrix} \end{pmatrix} d\xi\, d\theta,$$

$$\bar{F}_{wpn} = \iint (\alpha_{33wn}\beta_r\psi_s) d\xi\, d\theta,$$

$$\bar{F}_{wpn}^{bc} = \alpha_{22wn}^{bc}(\psi_s)|_0^{2\pi} \int \beta_r d\xi$$

$$+\alpha_{25wn}^{bc}(\beta_r)|_0^1 \int \psi_s d\theta + \alpha_{33wn}^{bc}(\psi_s)|_0^{2\pi} \int \beta_r d\xi,$$

$$\bar{F}_1 = \iint (\beta_r\psi_s) d\xi\, d\theta\,, \bar{F}_{we} = \bar{C}_1\bar{F}_e\bar{V}_{DC}^2\bar{F}_1,$$

$$\bar{F}_{Ha} = \bar{F}\iint (\beta_r\beta_s \cos\Omega\tau) d\xi d\theta\,,$$

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