Using Soft Sets for a Parametric Assessment of Problem Solving Skills

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Received: March 7, 2021. Revised: February 12, 2022. Accepted: March 11, 2022. Published: March 28, 2022.

***Abstract:* - Volumes of research have been written about problem solving, which is one of the most important components of the human cognition that affects our lives for ages. In this work soft sets are used as means for obtaining a qualitative assessment of problem solving skills and of Case-Based Reasoning systems' performance. The concept of soft set is a generalization of Zadeh's fuzzy sets introduced by Molodstov in 1999 as a new mathematical tool for dealing with the existing in real world uncertainty in a parametric manner.**

***Key-Words:* - Soft Sets, Fuzzy Sets and Logic, Parametric Assessment, Problem Solving (PS), Multidimensional PS Framework (MPSF), Case-Based Reasoning (CBR).**

I. INTRODUCTION

Quality is a desirable characteristic of all human activities. This makes assessment one of the most important components of the processes connected to the application of those activities. The present author has developed in earlier works several methods for assessing human-machine performance under fuzzy conditions, including the measurement of uncertainty in fuzzy systems, the use of the Center of Gravity (COG) defuzzification technique, the use of fuzzy or grey numbers, etc. All these methods have been reviewed in [1]. Here a method using soft sets is developed for the assessment of *Problem Solving (PS)* skills in a parametric manner. Such kind of methods are very useful when the assessment has qualitative rather than quantitative characteristics.

The motivation for writing this paper came from the fact that frequently the student assessment is performed using not numerical, but linguistic grades, like A, B, C, D, E, F and sometimes B-, B+, etc. Also, it is important and useful to assess the student PS skills at each step of the PS process, as those steps are described by standard PS models (see section 3).

The rest of the paper is formulated as follows: The definition of soft set and its connection to fuzzy sets are presented in the next section. The primary models for PS are exposed in section 3. The assessment method is developed in fourth section and the paper closes with the final conclusion and some hints for future research contained in fifth section.

II. FUZZY AND SOFT SETS

Until the middle of the 1960's *probability theory* used to be the unique tool in hands of the experts for dealing with the existing in real life and science situations of uncertainty. Probability, however, based on the principles of the bivalent logic, has been proved sufficient for tackling problems of uncertainty connected only to randomness, but not those connected to imprecision or to incomplete information of the given data.

The *fuzzy set theory*, introduced by Zadeh in 1965 [2], and the connected to it infinite-valued in the interval [0, 1] *fuzzy logic* [3] gave to scientists the opportunity to model under conditions of uncertainty which are vague or not precisely defined, thus succeeding to mathematically solve problems whose statements are expressed in the natural language. Through fuzzy logic the fuzzy terminology is translated by algorithmic procedures into numerical values, operations are performed upon those values and the outcomes are returned into natural language statements in a reliable manner [4].

Fuzzy systems are considered to be part of the wider class of *Soft Computing,* also including *probabilistic reasoning* and *neural networks*, the function of which is based on the function of biological networks [5]. One may say that neural networks and fuzzy systems try to emulate the operation of the human brain. The former concentrate on the structure of the human mind, i.e. the "hardware", and the latter concentrate on the "software" emulating human reasoning.

Let U be the universal set of the discourse. It is recalled that a fuzzy set A on U is defined with the help of its *membership function* m: $U \to [0,1]$ as the set of the ordered pairs

$$A = \{(x, m(x)): x \in U\} \quad (1)$$

The real number m(x) is called the *membership degree* of x in A**.** The greater is m(x), the more x satisfies the characteristic property of A. Many authors, for reasons of simplicity, identify a fuzzy set with its membership function.

A crisp subset A of U is a fuzzy set on U with membership function taking the values m(x)=1, if x belongs to A, and 0 otherwise. In other words, the concept of fuzzy set is an extension of the concept of the ordinary sets.

It is of worth noting that there is not any exact rule for defining the membership function of a fuzzy set. The methods used for this purpose are usually empirical or statistical and the definition is not unique depending on the personal goals of the observer. The only restriction about it is to be compatible to the common logic; otherwise the resulting fuzzy set does not give a reliable description of the corresponding real situation.

For example, defining the fuzzy set of the young people of a country one could consider as young all those being less than 30 years old and another all those being less than 40 years old. As a result they assign different membership degrees to people with ages below those two upper bounds.

For general facts on fuzzy sets, fuzzy logic and the connected to them uncertainty we refer to the chapters 4-7 of the book [6].

A lot of research has been carried out during the last 60 years for improving and extending the fuzzy set theory on the purpose of tackling more effectively the existing uncertainty in problems of science, technology and everyday life. Various generalizations of the concept of fuzzy set and relative theories have been developed like the type-2 fuzzy set, the intuitionistic fuzzy set, the neutrosophic set, the rough set, the grey system theory, etc. [7].

In 1999 Dmtri Molodstov, Professor at the Computing Center of the Russian Academy of Sciences in Moscow, proposed the notion of *soft set* as a new mathematical tool for dealing with the uncertainty in a parametric manner [8].

Let E be a set of parameters, let A be a subset of E and let f be a mapping of A into the set Δ(U) of all subsets of U. Then the soft set on U connected to A, denoted by (f, A), is defined as the set of the ordered pairs

$$(f, A) = \{(e, f(e)): e \in A\} \quad (2)$$

In other words, a soft set is a paramametrized family of subsets of U. Intuitively, it is "soft" because the boundary of the set depends on the parameters. For each e in A, f(e) is called the *value set* of e in (f, A), while f is called the *approximation function* of (f, A).

For example, let U= $\{H_1, H_2, H_3\}$ be a set of houses and let E = $\{e_1, e_2, e_3\}$ be the set of the parameters e_1=cheap, e_2= beautiful and e_3= expensive. Let us further assume that H_1, H_2 are the cheap and H_2, H_3 are the beautiful houses. Set A = $\{e_1, e_2\}$, then a mapping f: $A \to \Delta(U)$ is defined by $f(e_1)=\{H_1, H_2\}$, $f(e_2)=\{H_2, H_3\}$. Therefore, the soft set (f, A) on U is the set of the ordered pairs

$$(f, A) = \{(e_1, \{H_1, H_2\}), (e_2, \{H_2, H_3\})\} \quad (3)$$

A fuzzy set on U with membership function y = m(x) is a soft set on U of the form (f, [0, 1]), where $f(\alpha)=\{x \in U: m(x) \geq \alpha\}$ is the corresponding α – cut of the fuzzy set, for each α in [0. 1]. The concept of soft set is, therefore, a generalization of the concept of fuzzy set.

An important advantage of soft sets is that, by using the set of parameters E, they pass through the existing difficulty of defining properly the membership function of a fuzzy set. For general facts on soft sets we refer to [9]

The theory of soft sets has found many and important applications to several sectors of the human activity like decision making, parameter reduction, data clustering and data dealing with incompleteness, etc. [10]. One of the most important steps for the theory of soft sets was to define mappings on soft sets, which was achieved by A. Kharal and B. Ahmad and was applied to the problem of medical diagnosis in medical expert systems [11]. But fuzzy mathematics has also significantly developed at the theoretical level providing important insights even into branches of classical mathematics like algebra, analysis, geometry, topology etc. For example, one can extend the concept of topological space, the most general category of mathematical space, to fuzzy structures and in particular can define soft topological spaces and generalize the concepts of convergence, continuity and compactness within such kind of spaces [12].

III. MODELS FOR PROBLEM SOLVING

The importance of PS for human cognition and the evolution of our society has been recognized by authors, scientists and educators for centuries and volumes of research have been written about it. Perhaps Martinez's [13] definition carries the modern message about PS: "PS can be defined simply as the pursuit of a goal when the path to that goal is uncertain. In other words, it's what you do when you don't know what you're doing."

In [14, 15] we have examined the role of the problem in learning mathematics and we have attempted a review of the evolution of research on PS in mathematics education from the time of Polya until today.

Polya laid during the 50's and 60's the foundation for exploration in heuristics PS being the first who described them in a way that they could be taught. The failure of the introduction of the "New Mathematics" in school education placed the attention of specialists during the 80's on the use of the problem as a tool and motive to teach and understand better mathematics. A framework was created describing the PS process and reasons for success or failure in PS, which was depicted in Schoenfeld's *Expert Performance Model (EPM)* for PS [16]. The steps of the PS process in this model (see Figure 1) are the *analysis* (S_1) of the problem, the *design* (S_2) of the solution through the *exploration* (S_3), the *implementation* (S_4) of it and the *verification* (S_5).

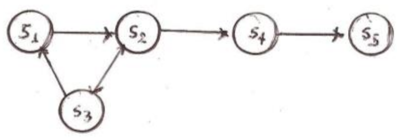

Figure 1: The "flow-diagram" of EPM

While early work on PS focused on describing the PS process, more recent investigations during the 2000's focused on identifying attributes of the problem solver that contribute to successful PS. Carlson and Bloom drawing from the large amount of literature related to PS developed a broad taxonomy to characterize major PS attributes that have been identified as relevant to PS success. This taxonomy gave genesis to their *Multidimensional PS Framework (MPSF)* [17], which includes the following steps: *Orientation, Planning, Executing* and *Checking*. It was observed that, when contemplating various solution approaches during the planning step of the PS process, the solvers were at times engaged in a *conjecture-imagine-evaluate* sub-cycle (see Figure 2).

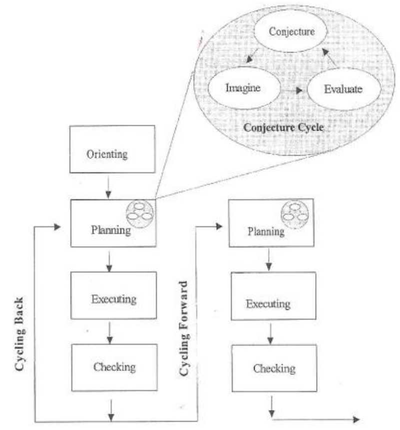

Figure 1: The "flow-diagram" of MPSF [17]

It is of worth noting that these two PS models share many similarities. In fact, a careful inspection shows that there exists a 1-1 correspondence between their steps, with S_1 corresponding to orientation, S_2 to planning, S_3 to the conjecture-imagine-evaluate sub-cycle, S_4 to executing and S_5 to checking. There is, however, a basic qualitative difference between the two models: While in MPSF the emphasis is turned to the solver's behavior and required attributes, the EPM is oriented towards the PS process itself describing the proper heuristic strategies that may be used at each step of the PS process.

IV. THE SOFT-SET ASSESSMENT MODEL

A. Assessment of Student PS skills

Assume that a teacher wants to assess the PS skills of a class of n students. Let $U = \{S_1, S_2, \ldots, S_n\}$ be the set of the students and let $E = \{e_1, e_2, e_3, e_4, e_5\}$ be the set of parameters e_1=excellent, e_2=very good, e_3=good, e_4=mediocre and e_5=unsatisfactory. Assume further that the first four students demonstrated excellent performance, the next five very good, the following 7 good, the next eight mediocre and the rest of them unsatisfactory performance. Let f be the map assigning to each parameter of E the subset of students whose performance was assessed by this parameter. Then, the overall student performance is represented mathematically by the soft set

$$(f, E) = \{(e_1, \{S_1, S_2, S_3\}), (e_2, \{S_4, S_5, \ldots, S_8\}), (e_3, \{S_9, S_{10}, \ldots, S_{15}\}), (e_4, \{S_{16}, S_{17}, \ldots, S_{23}\}), (e_5, \{S_{24}, S_{25}, \ldots, S_n\})\} \quad (4)$$

DOI: 10.46300/9106.2022.16.108

The use of soft sets enables also the representation of each student's individual performance at each step of the PS process. In fact, denote by S_1=orientation, S_2=planning, S_3=conjecture-imagine-evaluate, S_4=executing and S_5=checking the steps of the previously mentioned MPSF.

Set V = {S_1, S_2, S_3, S_4, S_5}, consider a particular student of the class and define a map f: $E \rightarrow \Delta(V)$ assigning to each parameter of E the subset of V consisting of the steps of the PS process assessed by this parameter with respect to the chosen student. For example, the soft set

$(f, E) = \{(e_1, \{S_1, S_3\}), (e_2, \{S_5\}), (e_3, \{S_4\}), (e_4, \{S_2\}), (e_5, \emptyset)\}$ (5)

represents the profile of a student who demonstrated excellent performance at the steps of orientation and conjecture-imagine-evaluate, very good performance at the step of checking, good performance at the step of executing and mediocre performance at the step of planning (he/she faced difficulties, but he/she finally came through).

B. Assessment of CBR Systems

One of the most popular PS strategies is the heuristic of the analogous problem. When the solver is not sure of the appropriate procedure to solve a given problem (referred as the *target problem*), a good hint would be to look for a similar problem solved in the past (*source problem*), and then try to adapt the solution procedure of the source problem for use with the target problem (*analogical PS*).

The important benefit of this strategy is that it precludes the necessity of constructing a new solution procedure. Using this strategy one has to specify it according to the form of the target problem; e.g. to solve a complex problem with many variables he/she may consider first an analogous problem with fewer variables, to solve a geometric problem in space he/she may consider first the corresponding problem in the plane, etc.

In a more general context *Case-Based Reasoning (CBR)* is the process of studying new cases based on the data of similar cases studied in the past [18]. The term PS is used in this case in a wide sense, coherent with common practice within the area of knowledge-based systems in general. This means that it is not necessarily the finding of a concrete solution to an application problem, it may be any problem put forth by the user. For example, to justify or criticize an already proposed solution, to interpret a problem situation, to generate a set of possible solutions, or generate explanations in observable data, are also PS situations.

CBR's coupling to learning occurs as a natural by-product of PS. When a problem is successfully solved, the experience is retained in order to solve similar problems in future. When an attempt to solve a problem fails, the reason for the failure is identified and remembered in order to avoid the same mistake in future. Thus CBR is a cyclic and integrated process of solving a problem, learning from this experience, solving a new problem, etc.

The CBR approach to PS and learning, for computers and people, has got a lot of attention over the last years, because as an intelligent-systems method enables information managers to increase efficiency and reduce cost by substantially automating processes such as diagnosis, scheduling, design, etc.

The use of computers enables the CBR systems to preserve a continuously increasing "library" of previously solved problems, referred as *past cases,* and to retrieve each time the suitable one for solving a given new problem. The CBR process involves the following steps:

- *Retrieve* (R_1) the most similar to the new problem past case, or cases.
- *Reuse* (R_2) the information and knowledge in that case to solve the new problem.
- *Revise* (R_3) the proposed solution.
- *Retain* (R_4) the parts of this experience likely to be useful for future problem-solving.

The quality of a CBR system can be assessed with the help of soft sets as follows:

Set U={R_1, R_2, R_3, R_4} and define a mapping f: $E \rightarrow \Delta(U)$ assigning to each parameter of E the subset of U consisting of the CBR steps whose quality was assessed by this parameter. For example, the soft set

$(f, E) = \{(e_1, \{R_1, R_4\}), (e_2, \{R_2\}), (e_3, \{R_3\}), (e_4, \emptyset), (e_5, \emptyset)\}$ (6)

corresponds to a CBR system which demonstrated excellent performance at the steps of retrieval and retaining of the past cases, very good performance at the step of reusing them and good performance in revising the selected past case for obtaining the solution of the new problem.

Also, given a set V of CBR systems, one can represent and compare their performance with a soft set of the form

$$(f, E) = \{(e_i, f(e_i)): i = 1, 2, 3, 4, 5\} \quad (7)$$

where f: $E \rightarrow \Delta(V)$ is a mapping assigning to each parameter of E the subset of V consisting of the CBR systems whose performance was assessed by this parameter.

V. CONCLUSION

The discussion performed in this work leads to the conclusion that soft sets offer a potential tool for a qualitative assessment of PS skills and of CBR systems' performance in a parametric manner. Due to its general texture, however, this soft-set assessment method could be also applied to a variety of other cases for assessing human or machine activities and this is an interesting subject for future research.

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